Matrix Mappings

Definition: A function f is a rule that assigns to every element x of a set called the domain a unique value y in another set called the codomain.

Examples

Let the domain and codomain be $\mathbb{R}$.

- The rule "x goes to $y = x^2$" is a function.
- The rule "x goes to $y = x + 6$" is a function.
- The rule "x goes to $y = 7$" is a function
- The rule "x goes to y such that $y^2 = x$" is not a function.

  The rule does not assign a y to every x. For example, there is no y value for $x = -1$.

  If we restrict the domain to the non-negative reals, the rule is still not a function, since it would not assign a **unique** value of y. For example, both $y = 2$ and $y = -2$ would be assigned to $x = 4$.

  If we restrict the domain and the codomain to the non-negative reals, then this rule is a function.

Notation

- If f is a function with domain U and codomain V, then we say that f maps U to V, and denote this by $f : U \to V$.
- We say that a function f maps x to y, or that y is the image of x under f, and we write $f(x) = y$.

Matrix Mappings

Example

Consider the function $f : \mathbb{R}^4 \to \mathbb{R}^2$, such that $f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$.

Then

$$f\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad f\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad f\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Matrix Mappings

Example

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

We first note that the product $\begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is in fact a 3×1 matrix, and thus is the same as an element of $\mathbb{R}^3$.

And we have that

$$f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 9 \end{bmatrix}$$

$$f\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Matrix Mappings

Definition: For any $m \times n$ matrix A, we define a function $f_A : \mathbb{R}^n \to \mathbb{R}^m$ called the matrix mapping corresponding to A by

$$f_A(\vec{x}) = A\vec{x} \qquad \text{for any } \vec{x} \in \mathbb{R}^n.$$

Note: Mostly just for purposes of typesetting, we sometimes write the vector $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ as the point $(x_1, x_2, \ldots, x_n)$.

Matrix Mappings

Example

Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Then the domain of f_A is $\mathbb{R}^4$ (since A has 4 columns) and the codomain of f_A is $\mathbb{R}^2$ (since A has 2 rows).

And we have that

$$f_A(1,2,3,4) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = (2,4)$$

$$f_A(4,-2,9,-1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \\ 9 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = (-2,-1)$$

$$f_A(1,0,0,0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (0,0)$$

$$f_A(0,1,0,0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1,0)$$

Matrix Mappings

Example

Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Notice that f_A is the same as the function $f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$ that was used in an earlier example.

We can prove this by noting that

$$f_A(x_1,x_2,x_3,x_4) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$$

for all $(x_1,x_2,x_3,x_4) \in \mathbb{R}^4$.