Matrix Multiplication

What really separates matrices from $\mathbb{R}^n$ is the fact that we define multiplication of matrices.

But note that matrix multiplication is *not* defined entrywise.

The inspiration for matrix multiplication comes from systems of linear equations.

Matrix Multiplication

Consider the following system:

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & 5x_3 & = & 2 \\ 2x_1 & & & + & 2x_3 & = & 0 \\ -x_1 & & & + & x_3 & = & -2 \\ -2x_1 & + & 4x_2 & - & 3x_3 & = & -7 \end{array}$$

The coefficient matrix for this system is

$$\begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & 2 \\ -1 & 0 & 1 \\ -2 & 4 & -3 \end{bmatrix}$$

Matrix multiplication is defined in such a way that we may rewrite our system using a product of matrices:

$$\begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & 2 \\ -1 & 0 & 1 \\ -2 & 4 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ -7 \end{bmatrix}$$

Note: Although we normally think of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -2 \\ -7 \end{bmatrix}$ as vectors, we can also think of them as 3×1 and 4×1 matrices respectively.

Matrix Multiplication

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Let's now see how we get from the matrix product to our system of equations.

First, the coefficients in each *row* of the coefficient matrix need to get matched up with the variables in the *column* of the second matrix.

Thinking of the rows of the coefficient matrix as vectors, we see that

$$\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 - 3x_2 + 5x_3 \qquad \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1 + 2x_3$$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -x_1 + x_3 \qquad \begin{bmatrix} -2 \\ 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -2x_1 + 4x_2 - 3x_3$$

These four expressions make up the left hand side of the equations in our system!

Matrix Multiplication

Thus, we want the matrix product $\begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & 2 \\ -1 & 0 & 1 \\ -2 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to be

$$\begin{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ \begin{bmatrix} -2 \\ 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 + 5x_3 \\ 2x_1 + 2x_3 \\ -x_1 + x_3 \\ -2x_1 + 4x_2 - 3x_3 \end{bmatrix}$$

The crucial idea here is that we take the dot product of a row with a column.

Matrix Multiplication

Definition: Let B be an $m \times n$ matrix with rows $\vec{b}_1^T, \dots, \vec{b}_m^T$, and let A be an $n \times p$ matrix with columns $\vec{a}_1, \dots, \vec{a}_p$. Then we define the matrix product BA to be the matrix whose ij-th entry is $(BA)_{ij} = \vec{b}_i \cdot \vec{a}_j$.

That is,

$$BA = \begin{bmatrix} \vec{b}_1^T \\ \vec{b}_2^T \\ \vdots \\ \vec{b}_i^T \\ \vdots \\ \vec{b}_m^T \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_j & \cdots & \vec{a}_p \end{bmatrix} = \begin{bmatrix} \vec{b}_1 \cdot \vec{a}_1 & \vec{b}_1 \cdot \vec{a}_2 & \cdots & \vec{b}_1 \cdot \vec{a}_j & \cdots & \vec{b}_1 \cdot \vec{a}_p \\ \vec{b}_2 \cdot \vec{a}_1 & \vec{b}_2 \cdot \vec{a}_2 & \cdots & \vec{b}_2 \cdot \vec{a}_j & \cdots & \vec{b}_2 \cdot \vec{a}_p \\ \vdots & \vdots & & \vdots & & \vdots \\ \vec{b}_i \cdot \vec{a}_1 & \vec{b}_i \cdot \vec{a}_2 & \cdots & \vec{b}_i \cdot \vec{a}_j & \cdots & \vec{b}_i \cdot \vec{a}_p \\ \vdots & \vdots & & \vdots & & \vdots \\ \vec{b}_m \cdot \vec{a}_1 & \vec{b}_m \cdot \vec{a}_2 & \cdots & \vec{b}_m \cdot \vec{a}_j & \cdots & \vec{b}_m \cdot \vec{a}_p \end{bmatrix}$$

Notes:

- We will end up taking the dot product of every row of B with every column of A.
- Every possible row-column combination will occur, and when we take the dot product of the i-th row of B with the j-th column of A, we end up with the ij-th entry of BA.

Matrix Multiplication

Example

$$\begin{bmatrix} 1 & 4 \\ -5 & 7 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 & 7 \\ -5 & 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -5 \end{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}$$

Matrix Multiplication

Example

$$\begin{bmatrix} 1 & 4 \\ -5 & 7 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 & 7 \\ -5 & 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -5 \end{bmatrix} & \begin{bmatrix} 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} & & \\ & & & \\ & & & \end{bmatrix}$$

Matrix Multiplication

Example

$$\begin{bmatrix} 1 & 4 \\ -5 & 7 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 & 7 \\ -5 & 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -5 \end{bmatrix} & \begin{bmatrix} 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} & \\ & & & \\ & & & \end{bmatrix}$$

Matrix Multiplication

Example

$$\begin{bmatrix} 1 & 4 \\ -5 & 7 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 & 7 \\ -5 & 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix}\cdot\begin{bmatrix} 2 \\ -5 \end{bmatrix} & \begin{bmatrix} 1 \\ 4 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 4 \end{bmatrix}\cdot\begin{bmatrix} 3 \\ 4 \end{bmatrix} & \begin{bmatrix} 1 \\ 4 \end{bmatrix}\cdot\begin{bmatrix} 7 \\ -1 \end{bmatrix} \\ & & & \\ & & & \end{bmatrix}$$

Matrix Multiplication

Example

$$\begin{bmatrix} 1 & 4 \\ -5 & 7 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 & 7 \\ -5 & 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix}\cdot\begin{bmatrix} 2 \\ -5 \end{bmatrix} & \begin{bmatrix} 1 \\ 4 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 4 \end{bmatrix}\cdot\begin{bmatrix} 3 \\ 4 \end{bmatrix} & \begin{bmatrix} 1 \\ 4 \end{bmatrix}\cdot\begin{bmatrix} 7 \\ -1 \end{bmatrix} \\ \begin{bmatrix} -5 \\ 7 \end{bmatrix}\cdot\begin{bmatrix} 2 \\ -5 \end{bmatrix} & \begin{bmatrix} -5 \\ 7 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \end{bmatrix} & \begin{bmatrix} -5 \\ 7 \end{bmatrix}\cdot\begin{bmatrix} 3 \\ 4 \end{bmatrix} & \begin{bmatrix} -5 \\ 7 \end{bmatrix}\cdot\begin{bmatrix} 7 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ -3 \end{bmatrix}\cdot\begin{bmatrix} 2 \\ -5 \end{bmatrix} & \begin{bmatrix} 0 \\ -3 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ -3 \end{bmatrix}\cdot\begin{bmatrix} 3 \\ 4 \end{bmatrix} & \begin{bmatrix} 0 \\ -3 \end{bmatrix}\cdot\begin{bmatrix} 7 \\ -1 \end{bmatrix} \end{bmatrix}$$

Matrix Multiplication

Example

$$\begin{bmatrix} 1 & 4 \\ -5 & 7 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 & 7 \\ -5 & 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix}1\\4\end{bmatrix}\cdot\begin{bmatrix}2\\-5\end{bmatrix} & \begin{bmatrix}1\\4\end{bmatrix}\cdot\begin{bmatrix}-1\\1\end{bmatrix} & \begin{bmatrix}1\\4\end{bmatrix}\cdot\begin{bmatrix}3\\4\end{bmatrix} & \begin{bmatrix}1\\4\end{bmatrix}\cdot\begin{bmatrix}7\\-1\end{bmatrix} \\ \begin{bmatrix}-5\\7\end{bmatrix}\cdot\begin{bmatrix}2\\-5\end{bmatrix} & \begin{bmatrix}-5\\7\end{bmatrix}\cdot\begin{bmatrix}-1\\1\end{bmatrix} & \begin{bmatrix}-5\\7\end{bmatrix}\cdot\begin{bmatrix}3\\4\end{bmatrix} & \begin{bmatrix}-5\\7\end{bmatrix}\cdot\begin{bmatrix}7\\-1\end{bmatrix} \\ \begin{bmatrix}0\\-3\end{bmatrix}\cdot\begin{bmatrix}2\\-5\end{bmatrix} & \begin{bmatrix}0\\-3\end{bmatrix}\cdot\begin{bmatrix}-1\\1\end{bmatrix} & \begin{bmatrix}0\\-3\end{bmatrix}\cdot\begin{bmatrix}3\\4\end{bmatrix} & \begin{bmatrix}0\\-3\end{bmatrix}\cdot\begin{bmatrix}7\\-1\end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -18 & 3 & 19 & 3 \\ -45 & 12 & 13 & -42 \\ 15 & -3 & -12 & 3 \end{bmatrix}$$

Matrix Multiplication

Notes on Matrix Multiplication

- In order to calculate the product BA, we need A and B to have *compatible sizes*. That is, the number of columns in B must be the same as the number of rows in A.
- If B is an $m \times n$ matrix, and A is an $n \times p$ matrix, then BA will be an $m \times p$ matrix.

Example

$$\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & -5 & 6 \\ 9 & -8 & 7 & 2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix}2\\3\end{bmatrix}\cdot\begin{bmatrix}1\\9\end{bmatrix} & \begin{bmatrix}2\\3\end{bmatrix}\cdot\begin{bmatrix}0\\-8\end{bmatrix} & \begin{bmatrix}2\\3\end{bmatrix}\cdot\begin{bmatrix}-5\\7\end{bmatrix} & \begin{bmatrix}2\\3\end{bmatrix}\cdot\begin{bmatrix}6\\2\end{bmatrix} \\ \begin{bmatrix}-1\\4\end{bmatrix}\cdot\begin{bmatrix}1\\9\end{bmatrix} & \begin{bmatrix}-1\\4\end{bmatrix}\cdot\begin{bmatrix}0\\-8\end{bmatrix} & \begin{bmatrix}-1\\4\end{bmatrix}\cdot\begin{bmatrix}-5\\7\end{bmatrix} & \begin{bmatrix}-1\\4\end{bmatrix}\cdot\begin{bmatrix}6\\2\end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -24 & 11 & 18 \\ 35 & -32 & 33 & 2 \end{bmatrix}$$

But the product $\begin{bmatrix} 1 & 0 & -5 & 6 \\ 9 & -8 & 7 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ does not exist, because the number of columns in the matrix on the left is not the same as the number of rows in the matrix on the right.

Matrix Multiplication

Notes on Matrix Multiplication

- In order to calculate the product BA, we need A and B to have *compatible sizes*. That is, the number of columns in B must be the same as the number of rows in A.
- If B is an $m \times n$ matrix, and A is an $n \times p$ matrix, then BA will be an $m \times p$ matrix.

Example

$$\begin{bmatrix} 1 & -1 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ -8 \end{bmatrix} = [(1)(2) + (-1)(-3) + (5)(-8)] = [-35]$$

However,

$$\begin{bmatrix} 2 \\ -3 \\ -8 \end{bmatrix} \begin{bmatrix} 1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} (2)(1) & (2)(-1) & (2)(5) \\ (-3)(1) & (-3)(-1) & (-3)(5) \\ (-8)(1) & (-8)(-1) & (-8)(5) \end{bmatrix} = \begin{bmatrix} 2 & -2 & 10 \\ -3 & 3 & -15 \\ -8 & 8 & -40 \end{bmatrix}$$

Matrix Multiplication

More Notes

Matrix multiplication is not commutative! That is, we do not always have that $AB = BA$.
The first example demonstrates that BA may not even be defined, even if AB is defined.
The second example demonstrates that even if both AB and BA are defined, they may not be the same size, let alone equal.
In any case (and even when BA and AB both exist and are the same size), there is no guarentee that $AB = BA$.

Example

$$\begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} (3)(1) + (0)(-1) & (3)(4) + (0)(-4) \\ (2)(1) + (0)(-1) & (2)(4) + (0)(-4) \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ 2 & 8 \end{bmatrix}$$

But we see that

$$\begin{bmatrix} 1 & 4 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} (1)(3) + (4)(2) & (1)(0) + (4)(0) \\ (-1)(3) + (-4)(2) & (-1)(0) + (-4)(0) \end{bmatrix} = \begin{bmatrix} 11 & 0 \\ -11 & 0 \end{bmatrix} \neq \begin{bmatrix} 3 & 12 \\ 2 & 8 \end{bmatrix}$$