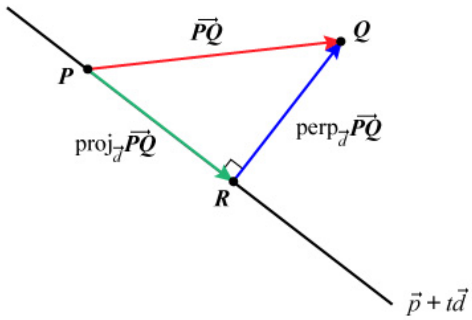
Minimum Distance

Definition: The distance from the line $\vec{x} = \vec{p} + t\vec{d}$ to the point Q is the minimum distance from the point Q to any point on the line, which equals $\|\text{perp}_{\vec{d}}\vec{PQ}\|$.



Let R be the point on the line $\vec{x} = \vec{p} + t\vec{d}$ that is closest to the point Q.
Then

$$\vec{r} = \vec{p} + \text{proj}_{\vec{d}}\vec{PQ} = \vec{q} - \text{perp}_{\vec{d}}\vec{PQ}$$

The result generalizes to $\mathbb{R}^n$.

Minimum Distance

Example

Find the point on the line $\vec{x} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$ that is closest to the point $Q(1, 2, 1)$, and calculate the distance from Q to the line.

Solution

First we note that P is $(0, 2, -2)$, and then we calculate that

$$\vec{PQ} = \vec{q} - \vec{p} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Next, to find $\text{proj}_{\vec{d}}\vec{PQ}$, we calculate

$$\vec{d} \cdot \vec{PQ} = (3)(1) + (0)(0) + (-3)(3) = -6$$
$$\|\vec{d}\|^2 = (3)^2 + (0)^2 + (-3)^2 = 18$$

This means that

$$\text{proj}_{\vec{d}}\vec{PQ} = \frac{\vec{d} \cdot \vec{PQ}}{\|\vec{d}\|^2}\vec{d} = \frac{-6}{18}\vec{d} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So $\vec{r} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$, and thus the point on the line closest to Q is $R(-1, 2, -1)$.

Minimum Distance

Example

Find the point on the line $\vec{x} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$ that is closest to the point $Q(1, 2, 1)$, and calculate the distance from Q to the line.

Solution

Continuing on, we get

$$\text{perp}_{\vec{d}}\vec{PQ} = \vec{PQ} - \text{proj}_{\vec{d}}\vec{PQ} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

So the distance from Q to the line is $\|\text{perp}_{\vec{d}}\vec{PQ}\| = \sqrt{2^2 + 0^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$.

Minimum Distance

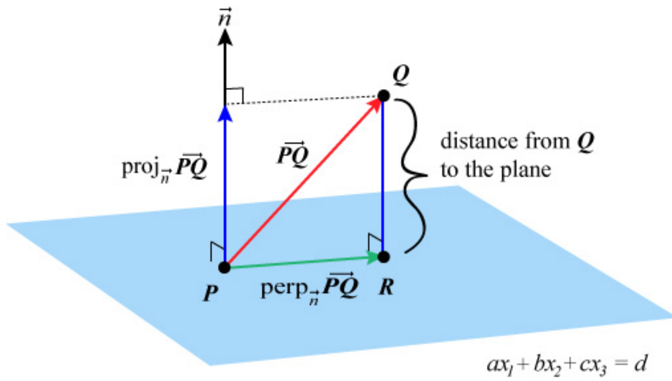
We can also look for the minimum distance from a point to a plane in $\mathbb{R}^3$.

We already know of one vector that is perpendicular to our plane: the normal vector.

So in this case we can compute the distance by looking at the projection of $\vec{PQ}$ onto the normal vector $\vec{n}$.

We can also use $\text{proj}_{\vec{n}}\vec{PQ}$ to find the point R on the plane closest to Q, although because our vector $\text{proj}_{\vec{n}}\vec{PQ}$ points out from the plane, we will need to multiply it by -1 to reverse its direction.

Minimum Distance



We see that

$$\vec{r} = \vec{q} - \text{proj}_{\vec{n}}\vec{PQ} = \vec{p} + \text{perp}_{\vec{n}}\vec{PQ}$$

and the distance from Q to the plane is

$$\|\text{proj}_{\vec{n}}\vec{PQ}\|$$

The text writes $\vec{OR} = \vec{OQ} + \text{proj}_{\vec{n}}\vec{QP}$, while I use the equation $\vec{r} = \vec{q} - \text{proj}_{\vec{n}}\vec{PQ}$.
Since $\text{proj}_{\vec{n}}\vec{QP} = \text{proj}_{\vec{n}}(-\vec{PQ}) = -\text{proj}_{\vec{n}}\vec{PQ}$ (by the linearity properties of proj), the two equations really are the same.

Minimum Distance

Example

Find the distance from the point $Q(1, 2, 1)$ to the plane $2x_1 + 3x_2 - x_3 = 4$, and find the point R on the plane closest to Q.

Solution

To do this, we first want to find a point P on the plane.
If we set $x_2 = x_3 = 0$, we get $2x_1 = 4 \Rightarrow x_1 = 2$.
So we see that $P(2, 0, 0)$ is a point on our plane.

This gives us $\vec{PQ} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

We can read off from the equation that $\vec{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

The next step is to find $\text{proj}_{\vec{n}}\vec{PQ}$, which we begin by calculating

$$\vec{n} \cdot \vec{PQ} = (2)(-1) + (3)(2) + (-1)(1) = 3$$
$$\|\vec{n}\|^2 = 2^2 + 3^2 + (-1)^2 = 14$$

So

$$\text{proj}_{\vec{n}}\vec{PQ} = \frac{\vec{n} \cdot \vec{PQ}}{\|\vec{n}\|^2}\vec{n} = \frac{3}{14}\vec{n} = \begin{bmatrix} 6/14 \\ 9/14 \\ -3/14 \end{bmatrix}$$

Minimum Distance

Example

Find the distance from the point $Q(1, 2, 1)$ to the plane $2x_1 + 3x_2 - x_3 = 4$, and find the point R on the plane closest to Q.

Solution

Thus, the distance from Q to the plane is

$$\|\text{proj}_{\vec{n}}\vec{PQ}\| = \sqrt{(6/14)^2 + (9/14)^2 + (-3/14)^2} = \sqrt{(126/14^2)} = \sqrt{(9/14)} = \frac{3}{\sqrt{14}}$$

Now, to find the point R, we simply compute

$$\vec{r} = \vec{q} - \text{proj}_{\vec{n}}\vec{PQ} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 6/14 \\ 9/14 \\ -3/14 \end{bmatrix} = \begin{bmatrix} 4/7 \\ 19/14 \\ 17/14 \end{bmatrix}$$

So the point on the plane closest to Q is $R(4/7, 19/14, 17/14)$.

Minimum Distance

Example

Find the point on the hyperplane $x_1 + x_2 - x_3 - x_4 = 2$ that is closest to the point $Q(0, 0, 0, 0)$, and calculate the distance from Q to the hyperplane.

Solution

First, we need to find a point P on the hyperplane.
By inspection, we notice that $1 + 1 - 0 - 0 = 2$, so $(1, 1, 0, 0)$ is a point on the hyperplane.

And since Q is $(0, 0, 0, 0)$, we have that $\vec{PQ} = -\vec{p} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$.

From the equation of our hyperplane, we read off $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$.

Thus,

$$\vec{n} \cdot \vec{PQ} = (1)(-1) + (1)(-1) + (-1)(0) + (-1)(0) = -2$$

Next, we compute

$$\|\vec{n}\|^2 = 1^2 + 1^2 + (-1)^2 + (-1)^2 = 4$$

Minimum Distance

Example

Find the point on the hyperplane $x_1 + x_2 - x_3 - x_4 = 2$ that is closest to the point $Q(0,0,0,0)$, and calculate the distance from Q to the hyperplane.

Solution

So $\text{proj}_{\vec{n}}\vec{PQ} = \frac{\vec{n} \cdot \vec{PQ}}{\|\vec{n}\|^2}\vec{n} = (-2/4)\vec{n} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$.

As such, the point on the hyperplane closest to Q is $\vec{q} - \text{proj}_{\vec{n}}\vec{PQ} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$, or in point form $(1/2, 1/2, -1/2, -1/2)$.

And the distance from Q to the hyperplane is $\|\text{proj}_{\vec{n}}\vec{PQ}\| = \sqrt{(-1/2)^2 + (-1/2)^2 + (1/2)^2 + (1/2)^2} = 1$.