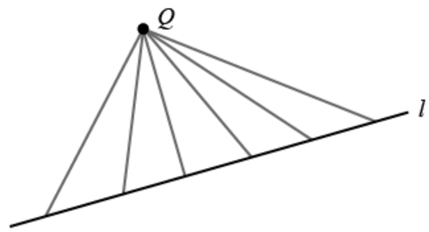
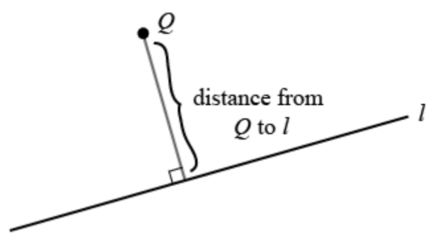
Projections and Perpendiculars



A common thing to look for is the distance from a point to a line.

That is, to look for the *shortest* distance from our starting point to a point on the line.

Projections and Perpendiculars

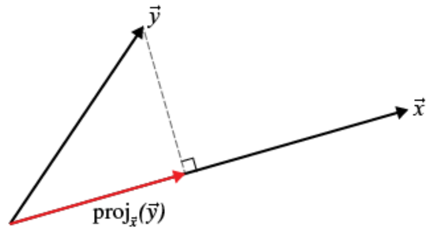


A common thing to look for is the distance from a point to a line.

That is, to look for the *shortest* distance from our starting point to a point on the line.

We assume that you know that we find the minimum distance by looking at the connecting segment that is perpendicular to our line.

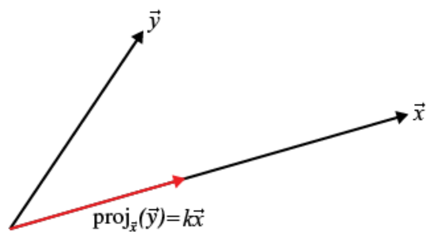
Projections and Perpendiculars



Definition: The part of $\vec{y}$ that is in the direction of $\vec{x}$ will be called the projection of $\vec{y}$ onto $\vec{x}$, and is denoted by $\text{proj}_{\vec{x}}(\vec{y})$.

Notice $\text{proj}_{\vec{x}}(\vec{y})$ will be in the same direction as $\vec{x}$, and thus will be a scalar multiple of $\vec{x}$.

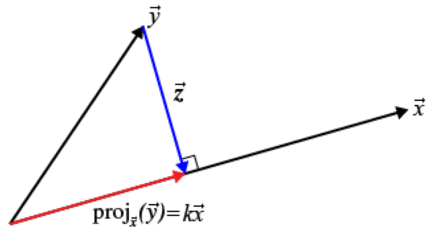
Projections and Perpendiculars



Definition: The part of $\vec{y}$ that is in the direction of $\vec{x}$ will be called the projection of $\vec{y}$ onto $\vec{x}$, and is denoted by $\text{proj}_{\vec{x}}(\vec{y})$.

Notice $\text{proj}_{\vec{x}}(\vec{y})$ will be in the same direction as $\vec{x}$, and thus will be a scalar multiple of $\vec{x}$.
Let k be the scalar, so that $\text{proj}_{\vec{x}}(\vec{y}) = k\vec{x}$.

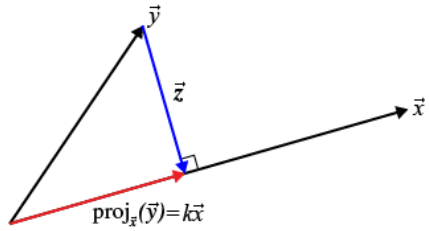
Projections and Perpendiculars



Definition: The part of $\vec{y}$ that is in the direction of $\vec{x}$ will be called the projection of $\vec{y}$ onto $\vec{x}$, and is denoted by $\mathrm{proj}_{\vec{x}}(\vec{y})$.

Notice $\mathrm{proj}_{\vec{x}}(\vec{y})$ will be in the same direction as $\vec{x}$, and thus will be a scalar multiple of $\vec{x}$.
Let k be the scalar, so that $\mathrm{proj}_{\vec{x}}(\vec{y}) = k\vec{x}$.
Now, let $\vec{z}$ be such that $\vec{y} + \vec{z} = \mathrm{proj}_{\vec{x}}(\vec{y})$.
Then $\vec{z}$ is orthogonal to $\vec{x}$.
Putting these facts together, we have that $k\vec{x} = \vec{y} + \vec{z}$.

Projections and Perpendiculars



Putting these facts together, we have that $k\vec{x} = \vec{y} + \vec{z}$.
We can take the dot product of both sides getting that $\vec{x} \cdot (k\vec{x}) = \vec{x} \cdot (\vec{y} + \vec{z}) \Rightarrow k(\vec{x} \cdot \vec{x}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$.
But since $\vec{z}$ is orthogonal to $\vec{x}$, we have $\vec{x} \cdot \vec{z} = 0$.
And therefore we get that $k = (\vec{x} \cdot \vec{y})/(\vec{x} \cdot \vec{x})$.
This gives us that

$$\mathrm{proj}_{\vec{x}}\vec{y} = \frac{\vec{y} \cdot \vec{x}}{||\vec{x}||^2}\,\vec{x}$$

Projections and Perpendiculars

Example

Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and let $\vec{y} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$.

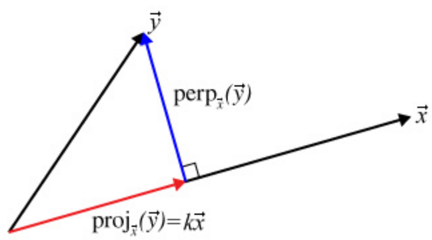
Then

$$\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x} = (1)(0) + (2)(3) + (-1)(4) = 2$$
$$||\vec{x}||^2 = 1^2 + 2^2 + (-1)^2 = 6$$
$$||\vec{y}||^2 = 0^2 + 3^2 + 4^2 = 25$$
$$\text{proj}_{\vec{x}}\vec{y} = (2/6)\vec{x} = (1/3)\vec{x} = \begin{bmatrix} 1/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$
$$\text{proj}_{\vec{y}}\vec{x} = (2/25)\vec{y} = \begin{bmatrix} 0 \\ 6/25 \\ 8/25 \end{bmatrix}$$

Projections and Perpendiculars

Definition: For any vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, with $\vec{x} \neq \vec{0}$, we define the projection of $\vec{y}$ perpendicular to $\vec{x}$ to be

$$\text{perp}_{\vec{x}}\vec{y} = \vec{y} - \text{proj}_{\vec{x}}\vec{y}$$



We see we can split a vector $\vec{y}$ into its portion in the direction of $\vec{x}$ and its portion perpendicular to $\vec{x}$, since $\vec{y} = \text{perp}_{\vec{x}}\vec{y} + \text{proj}_{\vec{x}}\vec{y}$.

Projections and Perpendiculars

Example

Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and let $\vec{y} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$.

Then in the previous example, we calculated that $\mathbf{proj}_{\vec{x}}\vec{y} = \begin{bmatrix} 1/3 \\ 2/3 \\ -1/3 \end{bmatrix}$ and $\mathbf{proj}_{\vec{y}}\vec{x} = \begin{bmatrix} 0 \\ 6/25 \\ 8/25 \end{bmatrix}$.

So now we get that

$$\mathbf{perp}_{\vec{x}}\vec{y} = \vec{y} - \mathbf{proj}_{\vec{x}}\vec{y} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 2/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 7/3 \\ 11/3 \end{bmatrix}$$

and

$$\mathbf{perp}_{\vec{y}}\vec{x} = \vec{x} - \mathbf{proj}_{\vec{y}}\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 6/25 \\ 8/25 \end{bmatrix} = \begin{bmatrix} 1 \\ 44/25 \\ -33/25 \end{bmatrix}$$

Linearity Properties of Projections

L1. $\text{proj}_{\vec{x}}(\vec{y} + \vec{z}) = \text{proj}_{\vec{x}}\vec{y} + \text{proj}_{\vec{x}}\vec{z}$ and $\text{perp}_{\vec{x}}(\vec{y} + \vec{z}) = \text{perp}_{\vec{x}}\vec{y} + \text{perp}_{\vec{x}}\vec{z}$ for all $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$

L2. $\text{proj}_{\vec{x}}(t\vec{y}) = t\ \text{proj}_{\vec{x}}\vec{y}$ and $\text{perp}_{\vec{x}}(t\vec{y}) = t\ \text{perp}_{\vec{x}}\vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$

Proof of L2

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

Then $\text{proj}_{\vec{x}}(t\vec{y}) = \dfrac{(t\vec{y}) \cdot \vec{x}}{||\vec{x}||^2}\,\vec{x} = \dfrac{t(\vec{y} \cdot \vec{x})}{||\vec{x}||^2}\,\vec{x} = t\,\dfrac{\vec{y} \cdot \vec{x}}{||\vec{x}||^2}\,\vec{x} = t\ \text{proj}_{\vec{x}}(\vec{y})$, as desired.

And this means that $\text{perp}_{\vec{x}}(t\vec{y}) = t\vec{y} - \text{proj}_{\vec{x}}(t\vec{y}) = t\vec{y} - t\ \text{proj}_{\vec{x}}(\vec{y}) = t(\vec{y} - \text{proj}_{\vec{x}}(\vec{y})) = t\ \text{perp}_{\vec{x}}\vec{y}$, as desired. □

The proof of **L1** is similar.

Note

- $\text{proj}_{\vec{x}}(\text{proj}_{\vec{x}}(\vec{y})) = \text{proj}_{\vec{x}}(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$
- $\text{perp}_{\vec{x}}(\text{perp}_{\vec{x}}(\vec{y})) = \text{perp}_{\vec{x}}(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$
- If $\vec{y}$ is a scalar multiple of $\vec{x}$, then $\text{proj}_{\vec{x}}(\vec{y}) = \vec{y}$ (and thus $\text{perp}_{\vec{x}}(\vec{y}) = \vec{0}$).
- If $\vec{y}$ is orthogonal to $\vec{x}$, then $\text{perp}_{\vec{x}}(\vec{y}) = \vec{y}$ (and thus $\text{proj}_{\vec{x}}(\vec{y}) = \vec{0}$).