Row Echelon Form

Recall that we want to develop an algorithm for solving a system of linear equations.

Definition: A matrix is in row echelon form (REF) if

1. When all entries in a row are zeros, this row appears below all rows that contain a non-zero entry.
2. When two non-zero rows are compared, the first non-zero entry (called the leading entry), in the upper row is to the left of the leading entry in the lower row.

Note that it follows from these properties that all entries in a column beneath a leading entry must be 0.

Definition: The leading entry in a non-zero row of a matrix in row echelon form is known as a pivot.

Row Echelon Form

Example

The matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 0 & -12 & -48 \\ 0 & 1 & 3 & 11 \end{array}\right]$$

is **not** in REF because the leading entry in the second row is not to the left of the leading entry in the third row.
We also note that there is a non-zero entry below the leading entry of the second row.
But,

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 0 & -12 & -48 \\ 0 & 1 & 3 & 11 \end{array}\right] R_2 \updownarrow R_3 \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & -12 & -48 \end{array}\right]$$

is in REF.

Example 2

The matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 4 & 13 \\ 0 & 0 & 1 & -3 & 8 \end{array}\right]$$

is in REF.

Row Echelon Form

These examples are taken from Lecture 2b to emphasize the connection between REF and the situation that we had in Lecture 2b where the elimination steps had been completed and all that was left were the back-substitution steps.

When solving linear systems, our first goal will be to row reduce our augmented matrix until it is in REF, and then we will translate our matrix back into a system of equations and use back-substitution to solve the system.

Row Echelon Form

Row-Reducing a Matrix to REF

Step 0: Is the matrix already in row echelon form?

If yes, you're done! If not, continue to the next step.

Step 1: Establish a pivot in R_1.

Identify the first **column** that contains a non-zero entry, and call it column A.
Interchanging rows if necessary, make the entry in column A of R_1 non-zero.
This will be the pivot for R_1.
Use EROs of type 3 to make all of the entries in column A below R_1 equal to zero.
It is convenient to make the pivot a 1.
If the matrix is now in REF, then you're done. Otherwise, continue to the next step.

Row Echelon Form

Row-Reducing a Matrix to REF

Example

Consider the matrix

$$\left[\begin{array}{ccccc|c} 0 & 4 & 8 & 0 & 3 & 0 \\ 3 & 7 & 17 & -3 & 2 & 2 \\ 1 & 2 & 5 & -1 & 0 & -1 \\ 5 & 11 & 27 & -5 & 4 & 8 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right]$$

Step 0: This matrix is not in row echelon form, since the leading entry in R_1 is not to the left of the leading entry in R_2.

Step 1: The first column contains non-zero entries, so we will interchange rows to make the entry in the first column of R_1 non-zero.

Seeing that the first entry in R_3 is a 1, we interchange R_1 and R_3, getting

$$\left[\begin{array}{ccccc|c} 0 & 4 & 8 & 0 & 3 & 0 \\ 3 & 7 & 17 & -3 & 2 & 2 \\ 1 & 2 & 5 & -1 & 0 & -1 \\ 5 & 11 & 27 & -5 & 4 & 8 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right] \begin{array}{c} R_1 \updownarrow R_3 \\ \\ \sim \\ \\ \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 3 & 7 & 17 & -3 & 2 & 2 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ 5 & 11 & 27 & -5 & 4 & 8 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right]$$

Row Echelon Form

Row-Reducing a Matrix to REF

Example

Consider the matrix

$$\left[\begin{array}{ccccc|c} 0 & 4 & 8 & 0 & 3 & 0 \\ 3 & 7 & 17 & -3 & 2 & 2 \\ 1 & 2 & 5 & -1 & 0 & -1 \\ 5 & 11 & 27 & -5 & 4 & 8 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right]$$

Step 0: This matrix is not in row echelon form, since the leading entry in R_1 is not to the left of the leading entry in R_2.

Step 1: The first column contains non-zero entries, so we will interchange rows to make the entry in the first column of R_1 non-zero.

Now we eliminate the 3 from the first entry of R_2

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 3 & 7 & 17 & -3 & 2 & 2 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ 5 & 11 & 27 & -5 & 4 & 8 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right] \begin{array}{c} \\ R_2 - 3R_1 \\ \sim \\ \\ \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ 5 & 11 & 27 & -5 & 4 & 8 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right]$$

Row Echelon Form

Row-Reducing a Matrix to REF

Example

Consider the matrix

$$\left[\begin{array}{ccccc|c} 0 & 4 & 8 & 0 & 3 & 0 \\ 3 & 7 & 17 & -3 & 2 & 2 \\ 1 & 2 & 5 & -1 & 0 & -1 \\ 5 & 11 & 27 & -5 & 4 & 8 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right]$$

Step 0: This matrix is not in row echelon form, since the leading entry in R_1 is not to the left of the leading entry in R_2.

Step 1: The first column contains non-zero entries, so we will interchange rows to make the entry in the first column of R_1 non-zero.

Next we eliminate the 5 from the first entry of R_4

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ 5 & 11 & 27 & -5 & 4 & 8 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right] R_4 - 5R_1 \sim \left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 4 & 13 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right]$$

Row Echelon Form

Row-Reducing a Matrix to REF

Example

Consider the matrix

$$\left[\begin{array}{ccccc|c} 0 & 4 & 8 & 0 & 3 & 0 \\ 3 & 7 & 17 & -3 & 2 & 2 \\ 1 & 2 & 5 & -1 & 0 & -1 \\ 5 & 11 & 27 & -5 & 4 & 8 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right]$$

Step 0: This matrix is not in row echelon form, since the leading entry in R_1 is not to the left of the leading entry in R_2.

Step 1: The first column contains non-zero entries, so we will interchange rows to make the entry in the first column of R_1 non-zero.

Finally, we eliminate -1 from the first entry of R_5

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 4 & 13 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right] R_5 + R_1 \sim \left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 4 & 13 \\ 0 & 3 & 6 & 0 & 6 & 3 \end{array}\right]$$

We have established a pivot in R_1, but the resulting matrix is not in REF since the leading entry in R_2 is not to the left of the leading entry in R_3.

Row Echelon Form

Row-Reducing a Matrix to REF

Step 2: Establish a pivot in R_2.

For Step 2 we will ignore R_1.
Let column B be the first column that has a non-zero entry in any row strictly below R_1.
Interchange rows if necessary to make the entry in column B of R_2 non-zero.
It is convenient to make the pivot a 1.
Use EROs of type 3 to make all of the entries in column B below R_2 equal to zero.
If the matrix is now in REF, then you're done. Otherwise, continue to the next step.

Row Echelon Form

Row-Reducing a Matrix to REF

Example (Continued)

After Step 1, we had that

$$\left[\begin{array}{ccccc|c} 0 & 4 & 8 & 0 & 3 & 0 \\ 3 & 7 & 17 & -3 & 2 & 2 \\ 1 & 2 & 5 & -1 & 0 & -1 \\ 5 & 11 & 27 & -5 & 4 & 8 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 4 & 13 \\ 0 & 3 & 6 & 0 & 6 & 3 \end{array}\right]$$

Step 2: If we ignore R_1, we see that the first column with a non-zero entry is column 2.
R_2 already has a non-zero entry in column 2, and it is a 1, so we do not need to interchange rows.
We remove the 4 from R_3, the 1 from R_4, and the 3 from R_5.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 4 & 13 \\ 0 & 3 & 6 & 0 & 6 & 3 \end{array}\right] \begin{array}{c} \\ \\ R_3 - 4R_2 \\ R_4 - R_2 \\ R_5 - 3R_2 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & -5 & -20 \\ 0 & 0 & 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{array}\right]$$

The resulting matrix is still not in REF since the leading entry in R_3 is not to the left of the leading entry in R_4. So we will move on to the next step.

Row Echelon Form

Row-Reducing a Matrix to REF

Step i: Establish a pivot in R_i.

By the time you reach step i, you will have established a pivot in each of rows 1 to $i-1$. We will ignore these rows.

Let column I be the first column that has a non-zero entry in a row below R_{i-1}.

Interchange rows if necessary to make the entry in column I of R_i non-zero.

Then use EROs of type 3 to make every entry in column I below R_i equal to zero.

If the matrix is now in REF, then you're done. Otherwise, proceed to Step $i+1$.

Row Echelon Form

Row-Reducing a Matrix to REF

Example (Continued)

Up to this point, we have row-reduced our orgininal matrix to the following matrix

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & -5 & -20 \\ 0 & 0 & 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{array}\right]$$

Step 3: We now ignore R_1 and R_2, and we see that the first column with a non-zero entry in the remaining rows is column 5.

R_3 already has a non-zero entry in column 5, and we see that if we multiply R_3 by $-1/5$, we can get the leading entry to be a 1.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & -5 & -20 \\ 0 & 0 & 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{array}\right] \begin{array}{c} \\ \\ \frac{1}{5}R_3 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{array}\right]$$

Row Echelon Form

Row-Reducing a Matrix to REF

Example (Continued)

Up to this point, we have row-reduced our orgininal matrix to the following matrix

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & -5 & -20 \\ 0 & 0 & 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{array}\right]$$

Step 3: We now ignore R_1 and R_2, and we see that the first column with a non-zero entry in the remaining rows is column 5.

To complete Step 3, we simply need to eliminate the **2** from R_4

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - 2R_3 \\ \\ \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{array}\right]$$

The matrix is still not in REF because R_4 is a row whose entries are all zeros, but it is above R_5, which is a row that contains a non-zero entry.

Row Echelon Form

Row-Reducing a Matrix to REF

Example (Continued)

Up to this point, we have row-reduced our orgininal matrix to the following matrix

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{array}\right]$$

Step 4: Establish a pivot in R_4.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 \updownarrow R_5 \\ \\ \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The matrix is finally in REF!

Row Echelon Form

More on this Example

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Looking at R_4 of our REF matrix, we see that it corresponds to a system containing the equation $0 = -12$.
Since this is never true, we know that our system of linear equations has no solutions.

It will be important to distinguish between the instruction solve the system of linear equations and obtain a row equivalent matrix in row echelon form.
If our goal had been to solve the system, we could have stopped after Step 3 and declared that there were no solutions.
However, for this particular example, the goal was to put a matrix in REF, so even after we saw that the corresponding system had no solutions, we needed to continue until we reached REF.

Row Echelon Form

Notes on Combining EROs

In Step 2, we saved time and space by performing three EROs "at the same time". We still wrote down the three calculations involved, but instead of changing the corresponding rows once per matrix, we jumped to the final matrix featuring the three new rows.
The rule to follow when deciding to do multiple operations at once is to make sure that you never change a specific row more than once, and that you never use a changing row to do an ERO of type 3 on another row.
In the previous example, while I subtracted a different multiple of R_2 from three different rows in Step 2, I did not actually alter R_2 at the same time.

Acceptable Combinations of EROs	Unacceptable Combinations of EROs
$R_1 \updownarrow R_2$ with $R_3 \updownarrow R_4$	$R_1 \updownarrow R_2$ with $R_3 \updownarrow R_2$
$2R_1$ with $2R_2$	$2R_1$ with $R_2 + 2R_1$

Note that we also do not want to combine steps like $2R_2$ with $R_2 + 3R_1$ into a step like $2R_2 + 3R_1$. Although $2R_2 + 3R_1$ does not cause confusion and results in a row equivalent matrix, it is **not** an ERO.