Solution Sets

Previously we noted that the solution to a homogenous system of linear equations is a subspace of $\mathbb{R}^n$.

If S is the solution set to the system of linear equations given by $A\vec{x} = \vec{b}$, where $\vec{b} \neq \vec{0}$, then S is **not a subspace** of $\mathbb{R}^n$.

The easiest way to see this is to note that $\vec{0} \notin S$, since $A\vec{0} = \vec{0}$, so $A\vec{0} \neq \vec{b}$.

Solution Sets

Theorem 3.4.2

Let $\vec{p}$ be a solution of the system of linear equations $A\vec{x} = \vec{b}$, $\vec{b} \neq \vec{0}$.

1. If $\vec{v}$ is any other solution of the same system, then $A(\vec{p} - \vec{v}) = \vec{0}$, so that $\vec{p} - \vec{v}$ is a solution of the corresponding homogeneous system $A\vec{x} = \vec{0}$.
2. If $\vec{h}$ is any solution of the corresponding system $A\vec{x} = \vec{0}$, then $\vec{p} + \vec{h}$ is a solution of the system $A\vec{x} = \vec{b}$.

The proof follows from the linearity properties of A, and is given in the textbook.

Solution Sets

Theorem 3.4.2 - Addition

Let $\vec{p}$ be a solution of the system of linear equations $A\vec{x} = \vec{b}$, $\vec{b} \neq \vec{0}$, and let $\text{Null}(A)$ be the solution set for the corresponding homogeneous system $A\vec{x} = \vec{0}$. Then the solution set for $A\vec{x} = \vec{b}$ is

$$S = \{\vec{x} + \vec{p} \mid \vec{x} \in \text{Null}(A)\}$$

Proof

To prove this fact, we need to show two things:

1. Every element of S (that is, every vector of the form $\vec{x} + \vec{p}$ where $\vec{x} \in \text{Null}(A)$) is a solution of $A\vec{x} = \vec{b}$, and
2. Every solution to $A\vec{x} = \vec{b}$ is in S, that is of the form $\vec{y} + \vec{p}$ where $\vec{y} \in \text{Null}(A)$.

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(1.) follows directly from part (2.) of Theorem 3.4.2.

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(1.) follows directly from part (2.) of Theorem 3.4.2.

For (2.), suppose $\vec{h}$ is a solution to $A\vec{x} = \vec{b}$.
Then Theorem 3.4.2 part (1.) says that $\vec{h} - \vec{p} \in \text{Null}(A)$.
So, let $\vec{h} - \vec{p}$ be our "$\vec{y}$", and we have that $\vec{h} = (\vec{h} - \vec{p}) + \vec{p} = \vec{y} + \vec{p}$, where $\vec{y} = (\vec{h} - \vec{p}) \in \text{Null}(A)$. $\square$

Solution Sets

Theorem 3.4.2 - Addition

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The addition to Theorem 3.4.2 says that the solution set of a non-homogeneous system is simply a translation of the solution set of the homogeneous system.
So, if the solution to the homogeneous system was a line through the origin, then the solution to the non-homogeneous system is a line through $\vec{p}$.
If the solution to the homogeneous system was a hyperplane through the origin, then the solution to the non-homogeneous system is a hyperplane through $\vec{p}$.
It also means that if you know that the general solution to $A\vec{x} = \vec{0}$ is $\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then the general solution to $A\vec{x} = \vec{b}$ would be $t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_n\vec{v}_n + \vec{p}$.