Solving a System of Linear Equations

Example

A woman has a coin purse that contains toonies, loonies, quarters, dimes, and nickels. The toonies and loonies have a combined value of $4, while the remaining coins have a combined value of $2.40. There are a total of 17 coins, and there is one more quarter than there are dimes. How many of each type of coin are in the coin purse?

Solution

First, we need to set this up as a system of linear equations.
Let x_1 be the number of nickels, x_2 be the number of dimes, x_3 be the number of quarters, x_4 be the number of loonies, and x_5 be the number of toonies.
The fact that she has a total of 17 coins becomes the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 17$$

The fact that there is one more quarter than there are dimes is expressed by the equation $x_3 = x_2 + 1$, but we will bring the x_2 over to the other side to put the equation into the standard form of a linear equation, getting

$$-x_2 + x_3 = 1$$

The fact that the toonies and loonies have a combined value of $4 gives us the following linear equation

$$x_4 + 2x_5 = 4$$

The fact that the quarters, dimes, and nickels have a combined value of $2.40 gives us the following linear equation

$$(.05)x_1 + (.1)x_2 + (.25)x_3 = 2.40$$

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Solution

We need to solve the following system of equations

$$\begin{array}{l} x_1 + x_2 + x_3 + x_4 + x_5 = 17 \\ -x_2 + x_3 = 1 \\ x_4 + 2x_5 = 4 \\ (.05)x_1 + (.1)x_2 + (.25)x_3 = 2.40 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ .05 & .1 & .25 & 0 & 0 & 2.40 \end{array}\right] \begin{array}{c} \\ (-1)R_2 \\ \\ 20R_4 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 1 & 2 & 5 & 0 & 0 & 48 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - R_1 \end{array} \sim$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 1 & 4 & -1 & -1 & 31 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - R_2 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 5 & -1 & -1 & 32 \end{array}\right] R_3 \updownarrow R_4 \sim$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 5 & -1 & -1 & 32 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{array}\right]$$

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$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 5 & -1 & -1 & 32 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{array}\right]$$

From the row echelon form, we see that that system is consistent and that the general solution will have one parameter.
The parameter will correspond to x_5, as column 5 does not contain a pivot.
Replacing the variable x_5 with the parameter s gives

$$\begin{aligned} &x_1 + x_2 + x_3 + x_4 + s = 17 \\ &x_2 - x_3 = -1 \\ &5x_3 - x_4 - s = 32 \\ &x_4 + 2s = 4 \end{aligned}$$

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We begin the back-substitution process by looking at the fourth equation

$$x_4 + 2s = 4 \Rightarrow x_4 = 4 - 2s$$

We substitute this value for x_4 into our third equation, getting

$$5x_3 - x_4 - s = 32 \Rightarrow 5x_3 - (4 - 2s) - s = 32 \Rightarrow 5x_3 = 36 - s \Rightarrow x_3 = (36/5) - (1/5)s$$

We substitute this value for x_3 into our second equation, getting

$$x_2 - x_3 = -1 \Rightarrow x_2 - ((36/5) - (1/5)s) = -1 \Rightarrow x_2 = (31/5) - (1/5)s$$

Finally, we substitute the above values of x_4, x_3, and x_2 into our first equation, getting

$$x_1 + x_2 + x_3 + x_4 + s = 17 \;\Rightarrow x_1 + (31/5) - (1/5)s + (36/5) - (1/5)s + 4 - 2s + s = 17$$
$$\Rightarrow x_1 = (-2/5) + (7/5)s$$

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Solution

From this, we see that the general solution to the system of linear equations is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} (-2/5)+(7/5)s \\ (31/5)-(1/5)s \\ (36/5)-(1/5)s \\ 4-2s \\ s \end{bmatrix} = \begin{bmatrix} -2/5 \\ 31/5 \\ 36/5 \\ 4 \\ 0 \end{bmatrix} + s \begin{bmatrix} 7/5 \\ -1/5 \\ -1/5 \\ -2 \\ 1 \end{bmatrix}$$

Recall that s represents the number of toonies, so it must be a non-negative integer.
Since the value of toonies and loonies is $\$4$, there can be at most two toonies.
Therefore, s is either $0, 1$ or 2.
We already saw s cannot be 0, since we cannot have $-2/5$ nickels.
Similarly if s is 2, then there would need to be $12/5$ nickels.
But when $s = 1$ we find that $x_1 = 1$, $x_2 = 6$, $x_3 = 7$, $x_4 = 2$, and $x_5 = 1$.
This gives us the only possible answer: that there are 1 nickel, 6 dimes, 7 quarters, 2 loonies, and 1 toonie in the coin purse.

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A Few Notes

- Math is more than just numbers and calculations.
- Once you have finished your calculations, make sure you go back and read the problem again to be sure you have answered the question.
- You need to understand **when** to plug something into a computer, **what** to plug in, **how to interpret** the results.