Span and Linear Independence

Definition: Let $\mathcal{B} = \{A_1, \dots, A_k\}$ be a set of $m \times n$ matrices, and let $t_1, \dots, t_k$ be real scalars. Then $t_1A_1 + t_2A_2 + \cdots + t_kA_k$ is a linear combination of the matrices in $\mathcal{B}$.

Definition: Let $\mathcal{B} = \{A_1, \dots, A_k\}$ be a set of $m \times n$ matrices. Then the span of $\mathcal{B}$ is defined as

$$\operatorname{Span}\mathcal{B} = \{t_1A_1 + \cdots + t_kA_k \mid t_1, \dots, t_k \in \mathbb{R}\}$$

That is, $\operatorname{Span}\mathcal{B}$ is the set of all linear combinations of the matrices in $\mathcal{B}$.

Definition: Let $\mathcal{B} = \{A_1, \dots, A_k\}$ be a set of $m \times n$ matrices. Then $\mathcal{B}$ is said to be linearly independent if the only solution to the equation

$$t_1A_1 + \cdots + t_kA_k = O_{m,n}$$

is the trivial solution $t_1 = \cdots = t_k = 0$. Otherwise, $\mathcal{B}$ is said to be linearly dependent.

Span and Linear Independence

Example

Determine if $\begin{bmatrix} 6 & 4 \\ 2 & 3 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 2 & 0 \\ 3 & -5 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ -9 & 8 \end{bmatrix} \right\}$.

Solution

To do this, we need to see if there are scalars t_1 and t_2 such that

$$t_1\begin{bmatrix} 2 & 0 \\ 3 & -5 \end{bmatrix} + t_2\begin{bmatrix} 4 & 4 \\ -9 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 3 \end{bmatrix}$$

Performing the operation on the left side, we see that we need

$$\begin{bmatrix} 2t_1 + 4t_2 & 4t_2 \\ 3t_1 - 9t_2 & -5t_1 + 8t_2 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 3 \end{bmatrix}$$

By the definition of matrix equality, we need t_1 and t_2 to be solutions to all of the following equations:

$$\begin{aligned} 2t_1 + 4t_2 &= 6 \\ 4t_2 &= 4 \\ 3t_1 - 9t_2 &= 2 \\ -5t_1 + 8t_2 &= 3 \end{aligned}$$

Span and Linear Independence

Example

Determine if $\begin{bmatrix} 6 & 4 \\ 2 & 3 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 2 & 0 \\ 3 & -5 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ -9 & 8 \end{bmatrix} \right\}$.

Solution

We can solve the system of linear equations by row reducing its augmented matrix:

$$\left[\begin{array}{cc|c} 2 & 4 & 6 \\ 0 & 4 & 4 \\ 3 & -9 & 2 \\ -5 & 8 & 3 \end{array}\right] \begin{array}{l} \frac{1}{2}R_1 \\ \frac{1}{4}R_2 \\ \\ \\ \end{array} \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 3 & -9 & 2 \\ -5 & 8 & 3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_1 \\ R_4 + 5R_1 \end{array} \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -15 & -7 \\ 0 & 18 & 18 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 15R_2 \\ R_4 - 18R_2 \end{array} \sim$$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{array}\right]$$

Since the third row in our REF matrix is a bad row, we see that the system has no solutions. This means that there are no t_1 and t_2 such that

$$t_1 \begin{bmatrix} 2 & 0 \\ 3 & -5 \end{bmatrix} + t_2 \begin{bmatrix} 4 & 4 \\ -9 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 3 \end{bmatrix}$$

Thus, $\begin{bmatrix} 6 & 4 \\ 2 & 3 \end{bmatrix}$ is not in the span of $\left\{ \begin{bmatrix} 2 & 0 \\ 3 & -5 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ -9 & 8 \end{bmatrix} \right\}$.

Span and Linear Independence

Example

Determine if $\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} \right\}$.

Solution

We need to see if there are scalars t_1, t_2, t_3, and t_4 such that

$$t_1 \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} + t_2 \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix} + t_3 \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix} + t_4 \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

Performing the operation on the left side, we see that we need

$$\begin{bmatrix} t_1 - 2t_3 + 2t_4 & t_1 + 3t_2 + 4t_3 + 2t_4 \\ 2t_1 + 4t_2 - 4t_3 - 4t_4 & 2t_1 - 3t_2 - 5t_3 + 3t_4 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

By the definition of equality, this means we are looking for solutions to the following system of linear equations:

$$\begin{array}{rcrcrcrcr} t_1 & & & - & 2t_3 & + & 2t_4 & = & -1 \\ t_1 & + & 3t_2 & + & 4t_3 & + & 2t_4 & = & 2 \\ 2t_1 & + & 4t_2 & - & 4t_3 & - & 4t_4 & = & 2 \\ 2t_1 & - & 3t_2 & - & 5t_3 & + & 3t_4 & = & 1 \end{array}$$

Span and Linear Independence

Example

Determine if $\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} \right\}$.

Solution

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 1 & 3 & 4 & 2 & 2 \\ 2 & 4 & -4 & -4 & 2 \\ 2 & -3 & -5 & 3 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - 2R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 3 & 6 & 0 & 3 \\ 0 & 4 & 0 & -8 & 4 \\ 0 & -3 & -1 & -1 & 3 \end{array}\right] \begin{array}{l} \frac{1}{3} R_2 \\ \frac{1}{4} R_3 \end{array} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & -3 & -1 & -1 & 3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \\ R_4 + 3R_2 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 5 & -1 & 6 \end{array}\right] -\frac{1}{2} R_3 \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 5 & -1 & 6 \end{array}\right] R_4 - 5R_3 \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -6 & 6 \end{array}\right] -\frac{1}{6} R_4 \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 - 2R_4 \\ \\ R_3 - R_4 \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 + 2R_3 \\ R_2 - 2R_3 \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

Span and Linear Independence

Example

Determine if $\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} \right\}$.

Solution

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 1 & 3 & 4 & 2 & 2 \\ 2 & 4 & -4 & -4 & 2 \\ 2 & -3 & -5 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

We see from the RREF matrix that $t_1 = 3$, $t_2 = -1$, $t_3 = 1$, $t_4 = -1$ is a solution to our system. This means that

$$3\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

Thus, $\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} \right\}$.

Span and Linear Independence

Example

Determine whether or not the set $\left\{ \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 8 & 6 \\ 5 & 1 \end{bmatrix} \right\}$ is linearly independent.

Solution

To do this, we need to see how many solutions there are to the equation

$$t_1 \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix} + t_2 \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix} + t_3 \begin{bmatrix} 8 & 6 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Performing the calculations on the left side, we see that this is the same as

$$\begin{bmatrix} t_1 + 8t_3 & 3t_1 - 2t_2 + 6t_3 \\ -t_1 + t_2 - 5t_3 & -3t_1 + 5t_2 - t_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This is the same as looking for solutions to the system of homogeneous equations

$$\begin{array}{rcrcrcl} t_1 & & & + & 8t_3 & = & 0 \\ 3t_1 & - & 2t_2 & + & 6t_3 & = & 0 \\ -t_1 & + & t_2 & - & 5t_3 & = & 0 \\ -3t_1 & + & 5t_2 & - & t_3 & = & 0 \end{array}$$

Span and Linear Independence

Example

Determine whether or not the set $\left\{ \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 8 & 6 \\ 5 & 1 \end{bmatrix} \right\}$ is linearly independent.

Solution

We solve this system by row reducing the coefficient matrix:

$$\begin{bmatrix} 1 & 0 & 8 \\ 3 & -2 & 6 \\ -1 & 1 & 5 \\ -3 & 5 & 1 \end{bmatrix} \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{array} \sim \begin{bmatrix} 1 & 0 & 8 \\ 0 & -2 & 18 \\ 0 & 1 & 13 \\ 0 & 5 & 25 \end{bmatrix} \begin{array}{l} \\ -\frac{1}{2} R_2 \\ \\ \frac{1}{5} R_4 \end{array} \sim \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -9 \\ 0 & 1 & 13 \\ 0 & 1 & 5 \end{bmatrix} \begin{array}{l} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{array} \sim$$

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -9 \\ 0 & 0 & 22 \\ 0 & 0 & 14 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_4 - \frac{14}{22} R_3 \end{array} \sim \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -9 \\ 0 & 0 & 22 \\ 0 & 0 & 0 \end{bmatrix}$$

From the final matrix being in row echelon form, we see that the rank of the coefficient matrix is 3.

Since this is the same as the number of variables, there are no parameters in the general solution to our homogeneous system. This means that there is only one solution to the system.

Thus, the set $\left\{ \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 8 & 6 \\ 5 & 1 \end{bmatrix} \right\}$ is linearly independent.

Span and Linear Independence

Example

Determine whether or not the set $\left\{\begin{bmatrix}1 & 1\\3 & -1\end{bmatrix}, \begin{bmatrix}-1 & 2\\-8 & 3\end{bmatrix}, \begin{bmatrix}2 & 11\\-9 & 4\end{bmatrix}\right\}$ is linearly independent.

Solution

To do this, we need to see how many solutions there are to the equation

$$t_1\begin{bmatrix}1 & 1\\3 & -1\end{bmatrix} + t_2\begin{bmatrix}-1 & 2\\-8 & 3\end{bmatrix} + t_3\begin{bmatrix}2 & 11\\-9 & 4\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

Performing the calculations on the left side, we see that this is the same as

$$\begin{bmatrix}t_1 - t_2 + 2t_3 & t_1 + 2t_2 + 11t_3\\3t_1 - 8t_2 - 9t_3 & -t_1 + 3t_2 + 4t_3\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

This is the same as looking for solutions to the system of homogeneous equations

$$\begin{array}{rcrcrcl} t_1 & - & t_2 & + & 2t_3 & = & 0\\ t_1 & + & 2t_2 & + & 11t_3 & = & 0\\ 3t_1 & - & 8t_2 & - & 9t_3 & = & 0\\ -t_1 & + & 3t_2 & + & 4t_3 & = & 0 \end{array}$$

Span and Linear Independence

Example

Determine whether or not the set $\left\{\begin{bmatrix}1 & 1\\3 & -1\end{bmatrix}, \begin{bmatrix}-1 & 2\\-8 & 3\end{bmatrix}, \begin{bmatrix}2 & 11\\-9 & 4\end{bmatrix}\right\}$ is linearly independent.

Solution

We solve this system by row reducing the coefficient matrix:

$$\begin{bmatrix}1 & -1 & 2\\1 & 2 & 11\\3 & -8 & -9\\-1 & 3 & 4\end{bmatrix} \begin{array}{c} \\ R_2 - R_1\\ R_3 - 3R_1\\ R_4 + R_1\end{array} \sim \begin{bmatrix}1 & -1 & 2\\0 & 3 & 9\\0 & -5 & -15\\0 & 2 & 6\end{bmatrix} \begin{array}{c}\frac{1}{3}R_2\\ -\frac{1}{5}R_3\\ \frac{1}{2}R_4\end{array} \sim \begin{bmatrix}1 & -1 & 2\\0 & 1 & 3\\0 & 1 & 3\\0 & 1 & 3\end{bmatrix} \begin{array}{c} \\ \\ R_3 - R_2\\ R_4 - R_2\end{array} \sim \begin{bmatrix}1 & -1 & 2\\0 & 1 & 3\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

This final matrix is in row echelon form, so we see that the rank of the coefficient matrix is 2.
Since the number of variables in the system is 3, this means that there are $3 - 2 = 1$ parameters in the general solution to the system.
Thus, $t_1 = t_2 = t_3 = 0$ is not the only solution to our equation, and this means that
$\left\{\begin{bmatrix}1 & 1\\3 & -1\end{bmatrix}, \begin{bmatrix}-1 & 2\\-8 & 3\end{bmatrix}, \begin{bmatrix}2 & 11\\-9 & 4\end{bmatrix}\right\}$ is linearly dependent. That is, it is **not** linearly independent.

Span and Linear Independence

One can continue the similarities with $\mathbb{R}^n$ by defining a subspace S of matrices to be a non-empty set of $m \times n$ matrices that satisfy properties (1) and (6) of Theorem 3.1.1.

We can also define a basis of such a subspace S to be a set $\mathcal{B}$ of $m \times n$ matrices that are both linearly independent and satisfying $\operatorname{Span} \mathcal{B} = S$.

We won't be considering matrix subspaces and bases in this course. Instead, we will now start to look at the properties of a matrix that are not extensions of our knowledge of $\mathbb{R}^n$.