Subspaces

Definition: A subset S of $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if the following conditions hold:

0. S is non-empty
1. S is closed under addition (that is, for $\vec{x}, \vec{y} \in S$ we have $\vec{x} + \vec{y} \in S$)
2. S is closed under scalar multiplication (that is, for $t \in \mathbb{R}$ and $\vec{x} \in S$, we have $t\vec{x} \in S$)

It is quick to show that $\{\vec{0}\}$ is always a subspace of $\mathbb{R}^n$, and $\mathbb{R}^n$ is always a subspace of $\mathbb{R}^n$.
With the exception of the set $\{\vec{0}\}$, subspaces will always contain an infinite number of elements. As such, we can't describe S with a list of elements.

Subspaces

Example 1

Consider the subset $S_1 = \{\vec{0}, \vec{e_1}, \vec{e_2}\}$ of $\mathbb{R}^2$.
Then S_1 is non-empty (and even contains $\vec{0}$).
But S_1 is not closed under addition, as $\vec{e_1} + \vec{e_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin S_1$.

S_1 is also not closed under scalar multiplication, as $2\vec{e_1} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, but $\begin{bmatrix} 2 \\ 0 \end{bmatrix} \notin S_1$.

So S_1 is **not** a subspace of $\mathbb{R}^2$.

Example 2

Consider the subset $S_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1 = 1 \right\}$ of $\mathbb{R}^2$.
Then $\vec{0} \notin S_2$, so S_2 is not a subspace of $\mathbb{R}^2$.

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Example 3

Consider the subset $S_3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1x_2 - x_3 = 0 \right\}$.

Then $\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \in S_3$ as $(2)(3) - 6 = 0$, but $2 \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 12 \end{bmatrix} \notin S_3$, as $(4)(6) - 12 = 12 \neq 0$.

So S_3 is not a subspace of $\mathbb{R}^3$.

Example 4

Consider the subset $S_4 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1, x_2 \leq 0 \right\}$ of $\mathbb{R}^2$.

Then $\begin{bmatrix} -1 \\ -1 \end{bmatrix} \in S_4$, but $(-1)\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin S_4$.

So S_4 is not a subspace of $\mathbb{R}^2$.

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Example 5

Consider the subset $S_5 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1 = 0 \right\}$ of $\mathbb{R}^2$.

Then $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in S_5$, so S_5 is non-empty.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be elements of S_5. Then $x_1 = y_1 = 0$.

Let $\vec{z} = \vec{x} + \vec{y}$.

Then $\vec{z} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} 0 + 0 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 + y_2 \end{bmatrix}$.

Since $z_1 = 0$ we see that $\vec{z} \in S_5$, and thus S_5 is closed under addition.

Finally, let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S_5$ and $t \in \mathbb{R}$, and let $\vec{z} = t\vec{x}$.

Then $\vec{z} = \begin{bmatrix} tx_1 \\ tx_2 \end{bmatrix} = \begin{bmatrix} t(0) \\ tx_2 \end{bmatrix} = \begin{bmatrix} 0 \\ tx_2 \end{bmatrix}$.

Since $z_1 = 0$ we have that $\vec{z} \in S_5$, and thus S_5 is closed under scalar multiplication.

And as S_5 is non-empty, closed under addition, and closed under scalar multiplication, we have that S_5 is a subspace of $\mathbb{R}^2$.

Subspaces

Example 6

Consider the subset $S_6 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_3 = x_1 + x_2 \right\}$ of $\mathbb{R}^3$.

Then $\vec{0} \in S_6$ since $0 = 0 + 0$. So S_6 is non-empty.

Next, suppose that $\vec{x}, \vec{y} \in S_6$. Then we have $x_3 = x_1 + x_2$ and $y_3 = y_1 + y_2$.
So let $\vec{z} = \vec{x} + \vec{y}$.
Then $z_1 = x_1 + y_1$, $z_2 = x_2 + y_2$, and
$z_3 = x_3 + y_3 = (x_1 + x_2) + (y_1 + y_2) = (x_1 + y_1) + (x_2 + y_2) = z_1 + z_2$.
So we have $\vec{z} \in S_6$, and thus that S_6 is closed under addition.

Finally, let $\vec{x} \in S_6$, $t \in \mathbb{R}$, and let $\vec{z} = t\vec{x}$.
Then $z_1 = tx_1$, $z_2 = tx_2$, and $z_3 = tx_3 = t(x_1 + x_2) = tx_1 + tx_2 = z_1 + z_2$.
So we see that $\vec{z} \in S_6$, and thus that S_6 is closed under scalar multiplication.

As we have shown that S_6 is non-empty, closed under addition, and closed under scalar multiplication, we have that S_6 is a subspace of $\mathbb{R}^3$.