## The Cofactor Matrix

**Definition:** Let $A$ be an $n \times n$ matrix. We define the cofactor matrix of $A$, denoted cof $A$, by

$$(\text{cof } A)_{ij} = C_{ij}$$

That is, the $ij$ entry of cof $A$ is the cofactor of the $ij$ entry of $A$.

**Note**

We have not looked at the cofactors of a $2 \times 2$ matrix.

To define the cofactor matrix of a $2 \times 2$ matrix we need to define the determinant of a $1 \times 1$ matrix.

The determinant of a $1 \times 1$ matrix $[a]$ is $a$.

This is consistent with Theorem 5.1.3, and with the definition of the determinant of a $2 \times 2$ matrix being the same as expanding by cofactors.

## The Cofactor Matrix

### Example

To determine the cofactor matrix of $A = \begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix}$, we first need to compute all the cofactors.

$C_{11} = (-1)^{1+1} \det\begin{bmatrix} -7 \end{bmatrix} = -7$

$C_{12} = (-1)^{1+2} \det\begin{bmatrix} 4 \end{bmatrix} = -4$

$C_{21} = (-1)^{2+1} \det\begin{bmatrix} 3 \end{bmatrix} = -3$

$C_{22} = (-1)^{2+2} \det\begin{bmatrix} 2 \end{bmatrix} = 2$

Note the use of the notation $\det\begin{bmatrix} -7 \end{bmatrix}$ instead of the abbreviated notation $\left| -7 \right|$ in this case, to prevent confusion with the idea of the absolute value.

And so we have

$$\text{cof } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} -7 & -4 \\ -3 & 2 \end{bmatrix}$$

## The Cofactor Matrix

### Example

To determine the cofactor matrix of $B = \begin{bmatrix} 7 & 1 & 3 \\ 4 & -2 & -5 \\ 9 & 8 & -3 \end{bmatrix}$, we first need to compute all the cofactors.

$C_{11} = (-1)^{1+1}\begin{vmatrix} -2 & -5 \\ 8 & -3 \end{vmatrix} = 6 + 40 = 46$ $\qquad$ $C_{12} = (-1)^{1+2}\begin{vmatrix} 4 & -5 \\ 9 & -3 \end{vmatrix} = -(-12 + 45) = -33$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 4 & -2 \\ 9 & 8 \end{vmatrix} = 32 + 18 = 50$ $\qquad$ $C_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 3 \\ 8 & -3 \end{vmatrix} = -(-3 - 24) = 27$

$C_{22} = (-1)^{2+2}\begin{vmatrix} 7 & 3 \\ 9 & -3 \end{vmatrix} = -21 - 27 = -48$ $\qquad$ $C_{23} = (-1)^{2+3}\begin{vmatrix} 7 & 1 \\ 9 & 8 \end{vmatrix} = -(56 - 9) = -47$

$C_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 3 \\ -2 & -5 \end{vmatrix} = -5 + 6 = 1$ $\qquad$ $C_{32} = (-1)^{3+2}\begin{vmatrix} 7 & 3 \\ 4 & -5 \end{vmatrix} = -(-35 - 12) = 47$

$C_{33} = (-1)^{3+3}\begin{vmatrix} 7 & 1 \\ 4 & -2 \end{vmatrix} = -14 - 4 = -18$

And so we have

$$\operatorname{cof} B = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 46 & -33 & 50 \\ 27 & -48 & -47 \\ 1 & 47 & -18 \end{bmatrix}$$