The Length of the Cross Product

Theorem 1.5.2

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$ and θ be the angle between $\vec{u}$ and $\vec{v}$. Then $||\vec{u} \times \vec{v}|| = ||\vec{u}||\ ||\vec{v}||\ \sin\theta$.

Consequence of Theorem 1.5.2

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$. Then the area of the parallelogram with base $\vec{u}$ and side $\vec{v}$ is $||\vec{u} \times \vec{v}||$.

Example

Find the area of the parallelogram determined by $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$.

Solution

First, we calculate that $\vec{u} \times \vec{v} = \begin{bmatrix} (1)(3) - (2)(4) \\ (2)(-1) - (2)(3) \\ (2)(4) - (1)(-1) \end{bmatrix} = \begin{bmatrix} -5 \\ -8 \\ 9 \end{bmatrix}$.

Then the area of the parallelogram is $||\vec{u} \times \vec{v}|| = \sqrt{(-5)^2 + (-8)^2 + 9^2} = \sqrt{170}$.