The Scalar Equation of a Hyperplane

We will develop the general scalar equation of a hyperplane by first focusing on the equation of a hyperplane in $\mathbb{R}^3$. Remember a hyperplane in $\mathbb{R}^3$ is the same as a plane.

Previously we saw the vector equation of a plane in $\mathbb{R}^3$ is $\vec{x} = \vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2$, where $\vec{p}$, $\vec{v}_1$, and $\vec{v}_2$ are all fixed vectors, and as we vary the scalars t_1 and t_2 we get all the vectors $\vec{x}$ on the plane.

But it happens that we can also get a scalar equation for a plane, of the form $ax_1 + bx_2 + cx_3 = d$, where $a, b, c, d \in \mathbb{R}$, and the vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is on the plane if and only if x_1, x_2, and x_3 satisfy the equation.

Since a hyperplane is one dimension smaller than the space it lives in, instead of listing the $n-1$ directions that are in the hyperplane, we look at the one direction that is **not** in the hyperplane.

So in $\mathbb{R}^3$, instead of describing a plane using the two direction vectors $\vec{v}_1$ and $\vec{v}_2$ that are in the plane, we are going to look for a third vector $\vec{n}$ that is not in the plane.

The vector we are looking for is called the normal vector for the plane, and it has the property that it is orthogonal to any directed line segment in the plane.

So, if $\vec{x}$ is on the plane $\vec{x} = \vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2$, then the directed line segment $\overrightarrow{PX} = \vec{x} - \vec{p}$ is in the plane, and so we want to satisify $\vec{n} \cdot (\vec{p} - \vec{x}) = 0$.

This means that $n_1(x_1 - p_1) + n_2(x_2 - p_2) + n_3(x_3 - p_3) = 0$, or that $n_1x_1 + n_2x_2 + n_3x_3 = n_1p_1 + n_2p_2 + n_3p_3$.

This means that, in our general scalar equation of a plane ($ax_1 + bx_2 + cx_3 = d$), we have $a = n_1$, $b = n_2$, $c = n_3$, and $d = \vec{n} \cdot \vec{p}$.

The Scalar Equation of a Hyperplane

Example

Find the scalar equation of the plane in $\mathbb{R}^3$ that passes through the point $P(-3, 2, 7)$ and has normal vector $\vec{n} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Solution

First we compute $\vec{n} \cdot \vec{p} = (3)(-3) + (2)(2) + (1)(7) = 2$.

Then the equation is $3x_1 + 2x_2 + x_3 = 2$.

The Scalar Equation of a Hyperplane

Definition: Two planes in $\mathbb{R}^3$ are defined to be parallel if the normal vector to one plane is a non-zero scalar multiple of the normal vector of the other plane.

Example

The plane $3x_1 + 2x_2 + x_3 = 2$ is parallel to the plane $9x_1 + 6x_2 + 3x_3 = -5$, (because $3\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}9\\6\\3\end{bmatrix}$).

However, the plane $3x_1 + 2x_2 + x_3 = 2$ is not parallel to $3x_1 + 2x_2 - x_3 = 7$ (since $\begin{bmatrix}3\\2\\1\end{bmatrix} \neq t\begin{bmatrix}3\\2\\-1\end{bmatrix}$ for any $t \in \mathbb{R}$).

The Scalar Equation of a Hyperplane

Example

Consider the planes $A = 3x_1 + 2x_2 + x_3 = 2$ and $B = 9x_1 + 6x_2 + 3x_3 = -5$.
We saw in the previous example that they are parallel.
They are not the same plane, as $(0, 1, 0)$ is on plane A (since $3(0) + 2(1) + 1(0) = 2$), but it is not on plane B (since $9(0) + 6(1) + 3(0) = 6 \neq -5$).
So, let (a, b, c) be a generic point on A.
Then we have $3a + 2b + c = 2$, which means that $9a + 6b + 3c = 3(3a + 2b + c) = 3(2) = 6 \neq -5$, so (a, b, c) is not on B.
Similarly, if we look at a generic point (a', b', c') on B, then it satisfies $9a' + 6b' + 3c' = -5$.
But this means that $3a' + 2b' + c' = (1/3)(9a' + 6b' + 3c') = (1/3)(-5) = -5/3 \neq 2$, so (a', b', c') is not on A.
As such, we see that A and B do not have any points in common.

The Scalar Equation of a Hyperplane

Definition: We say that two planes are orthogonal to each other if their normal vectors are orthogonal to each other.

Example

Consider the planes $3x_1 + 2x_2 + x_3 = 2$ and $-2x_1 + 4x_2 - 2x_3 = -3$.

They are orthogonal, because $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = (3)(-2) + (2)(4) + (1)(-2) = 0$.

The Scalar Equation of a Hyperplane

Let A be a hyperplane in $\mathbb{R}^n$ that contains the point $\vec{p}$, and let $\vec{m}$ be a vector that is orthogonal to every directed line segment in A.

Then the scalar equation for A is

$$m_1x_1 + m_2x_2 + \cdots + m_nx_n = \vec{m} \cdot \vec{p}$$

Example

Find the scalar equation of the hyperplane in $\mathbb{R}^5$ that has normal vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ and passes through the point $P(-1, 1, 4, 2, -4)$.

Solution

First we compute $\vec{m} \cdot \vec{p} = (1)(-1) + (2)(1) + (3)(4) + (4)(2) + (5)(-4) = 1$.

Then the equation for the hyperplane is $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 1$.