The Transpose of a Matrix

Definition: Let A be an $m \times n$ matrix. Then the transpose of A is the $n \times m$ matrix A^T whose ij-th entry is the ji-th entry of A. That is,

$$(A^T)_{ij} = (A)_{ji}$$

Examples

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -4 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \\ 4 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 8 \end{bmatrix}^T = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

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Properties of the Transpose

One interesting property of the transpose is that it is an operation on a matrix that changes the size of the matrix. Therefore, in general, $A + A^T$ will not be defined, nor will $A = A^T$ be true.
However, in the case that A is a sqaure matrix, $A + A^T$ is defined, and there is a possibility that $A = A^T$ (but this is not necessarily true).

Theorem 3.1.2

For any matrices A and B and scalar $s \in \mathbb{R}$, we have that

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(sA)^T = sA^T$

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Example

Let $A = \begin{bmatrix} -3 & 0 \\ 9 & 2 \\ -2 & 2 \\ 7 & -1 \end{bmatrix}$. Then $A^T = \begin{bmatrix} -3 & 9 & -2 & 7 \\ 0 & 2 & 2 & -1 \end{bmatrix}$

We see that

$$(A^T)^T = \begin{bmatrix} -3 & 0 \\ 9 & 2 \\ -2 & 2 \\ 7 & -1 \end{bmatrix} = A$$

Moreover, we note that

$$(5A)^T = \begin{bmatrix} -15 & 0 \\ 45 & 10 \\ -10 & 10 \\ 35 & -5 \end{bmatrix}^T = \begin{bmatrix} -15 & 45 & -10 & 35 \\ 0 & 10 & 10 & -5 \end{bmatrix} = 5\begin{bmatrix} -3 & 9 & -2 & 7 \\ 0 & 2 & 2 & -1 \end{bmatrix} = 5A^T$$

The Transpose of a Matrix

Example

Let $A = \begin{bmatrix} -3 & 0 \\ 9 & 2 \\ -2 & 2 \\ 7 & -1 \end{bmatrix}$. Then $A^T = \begin{bmatrix} -3 & 9 & -2 & 7 \\ 0 & 2 & 2 & -1 \end{bmatrix}$

Now, if $B = \begin{bmatrix} 4 & 1 \\ -5 & -3 \\ 0 & 2 \\ -9 & 6 \end{bmatrix}$, then $B^T = \begin{bmatrix} 4 & -5 & 0 & -9 \\ 1 & -3 & 2 & 6 \end{bmatrix}$.

Therefore,

$$A + B = \begin{bmatrix} -3 & 0 \\ 9 & 2 \\ -2 & 2 \\ 7 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ -5 & -3 \\ 0 & 2 \\ -9 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ -2 & 4 \\ -2 & 5 \end{bmatrix}$$

Hence, we see that

$$A^T + B^T = \begin{bmatrix} -3 & 9 & -2 & 7 \\ 0 & 2 & 2 & -1 \end{bmatrix} + \begin{bmatrix} 4 & -5 & 0 & -9 \\ 1 & -3 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -2 & -2 \\ 1 & -1 & 4 & 5 \end{bmatrix} = (A + B)^T$$