Vector Equation of a Line in $\mathbb{R}^3$

We still need a point and a direction vector to create a line in $\mathbb{R}^3$, as we did in $\mathbb{R}^2$.
There is no scalar form of the equation of a line in $\mathbb{R}^3$.
We can describe a line in $\mathbb{R}^3$ with parametric equations of the form

$$\begin{aligned} x_1 &= a_1 + tb_1 \\ x_2 &= a_2 + tb_2 \qquad t \in \mathbb{R} \\ x_3 &= a_3 + tb_3 \end{aligned}$$

The values a_1, a_2, and a_3 correspond to a point on the line.
The values b_1, b_2, and b_3 correspond to the direction the line travels in the x_1, x_2, and x_3 directions.
So, we can combine the three parametric equations to form the single vector equation

$$\vec{x} = \vec{a} + t\vec{b}, t \in \mathbb{R}$$

where $\vec{a}$ is a point on the line, and $\vec{b}$ is a direction vector for the line.

Example

Find a vector equation of the line that passes through the point $P(-3, -5, 1)$ with direction vector $\vec{d} = \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix}$.

We get that the equation is $\vec{x} = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} + t \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix}, t \in \mathbb{R}$.

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We are often trying to find the equation of a line through two points, say $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$.
When creating the parametric equation we compute that $b_1 = q_1 - p_1$, $b_2 = q_2 - p_2$, and $b_3 = q_3 - p_3$.
This means that our direction vector is $\vec{b} = \begin{bmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{bmatrix} = \vec{PQ}$.
So, a vector equation of the line through P and Q is

$$\vec{x} = \vec{p} + t\vec{PQ},\ t \in \mathbb{R}$$

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Example

Find a vector equation for the line that passes through $P(-5, 2, 10)$ and $Q(3, -4, -4)$.

Solution

First we calculate $\vec{PQ} = \begin{bmatrix} 3 \\ -4 \\ -4 \end{bmatrix} - \begin{bmatrix} -5 \\ 2 \\ 10 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \\ -14 \end{bmatrix}$.

Then we get that an equation for our line is

$$\vec{x} = \begin{bmatrix} -5 \\ 2 \\ 10 \end{bmatrix} + t \begin{bmatrix} 8 \\ -6 \\ -14 \end{bmatrix}, t \in \mathbb{R}$$

Pulling out a factor of 2 from the direction vector gives us

$$\vec{x} = \begin{bmatrix} -5 \\ 2 \\ 10 \end{bmatrix} + t \begin{bmatrix} 4 \\ -3 \\ -7 \end{bmatrix}, t \in \mathbb{R}$$

Using the point Q instead of P gives

$$\vec{x} = \begin{bmatrix} 3 \\ -4 \\ -4 \end{bmatrix} + t \begin{bmatrix} 4 \\ -3 \\ -7 \end{bmatrix}, t \in \mathbb{R}$$

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To check your answer is the same as the provided solution, pick two points on the provided line and make sure they are on your line.