## Linear Mappings Over $\mathbb{C}$

**Definition:** If $\mathbb{V}$ and $\mathbb{W}$ are vector spaces over the complex numbers, then a mapping $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping if for any $\alpha \in \mathbb{C}$ and $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{V}$ we have

$$L(\alpha\mathbf{v}_1 + \mathbf{v}_2) = \alpha L(\mathbf{v}_1) + L(\mathbf{v}_2)$$

### Example

The mapping $L : \mathbb{C}^4 \to \mathbb{C}^2$ defined by $L(z_1, z_2, z_3, z_4) = (2z_1 + 3z_2, 2iz_3 + 3iz_4)$ is a linear mapping. We prove this as follows:

$$\begin{aligned}
L(\alpha\vec{z} + \vec{w}) &= L(\alpha z_1 + w_1, \alpha z_2 + w_2, \alpha z_3 + w_3, \alpha z_4 + w_4) \\
&= (2(\alpha z_1 + w_1) + 3(\alpha z_2 + w_2), 2i(\alpha z_3 + w_3) + 3i(\alpha z_4 + w_4)) \\
&= (2\alpha z_1 + 2w_1 + 3\alpha z_2 + 3w_2, 2i\alpha z_3 + 2iw_3 + 3i\alpha z_4 + 3iw_4) \\
&= (2\alpha z_1 + 3\alpha z_2, 2i\alpha z_3 + 3i\alpha z_4) + (2w_1 + 3w_2, 2iw_3 + 3iw_4) \\
&= \alpha(2z_1 + 3z_2, 2iz_3 + 3iz_4) + (2w_1 + 3w_2, 2iw_3 + 3iw_4) \\
&= \alpha L(\vec{z}) + L(\vec{w})
\end{aligned}$$

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### Example

The mapping $M : \mathbb{C}^1 \to \mathbb{C}^1$ defined by $M(z) = \bar{z}$ is not a linear mapping, as it does not preserve scalar multiplication.

For example, if we let $\alpha = 1 - 3i$, $\mathbf{v}_1 = 2 + 4i$, and $\mathbf{v}_2 = 0$, then we see that
$M(\alpha\mathbf{v}_1 + \mathbf{v}_2) = M((1 - 3i)(2 + 4i) + 0) = M(2 + 4i - 6i - 12i^2) = M(14 - 2i) = 14 + 2i$

But $\alpha M(\mathbf{v}_1) + M(\mathbf{v}_2) = (1 - 3i)M(2 + 4i) + M(0) = (1 - 3i)(2 - 4i) + 0 = 2 - 4i - 6i + 12i^2 = -10 - 10i$

And so we have that $M(\alpha\mathbf{v}_1 + \mathbf{v}_2) \neq \alpha M(\mathbf{v}_1) + M(\mathbf{v}_2)$.

As in $\mathbb{R}^n$, we find that every linear mapping in $\mathbb{C}^n$ can be thought of as a matrix mapping.

### Example

$$L(z_1, z_2, z_3, z_4) = (2z_1 + 3z_2, 2iz_3 + 3iz_4) = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 0 & 2i & 3i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

In general, we know that if $z \in \mathbb{C}^n$, then we can write

$$z = z_1\vec{e}_1 + z_2\vec{e}_2 + \cdots + z_n\vec{e}_n$$

where $\vec{e}_i$ are the same as in $\mathbb{R}^n$. (Remember, every real number is a complex number, so every real vector is a complex vector.)

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### Example

This means we can write:

$$\begin{aligned} L(\vec{z}) &= L(z_1\vec{e}_1 + z_2\vec{e}_2 + \ldots + z_n\vec{e}_n) \\ &= z_1 L(\vec{e}_1) + z_2 L(\vec{e}_2) + \cdots + z_n L(\vec{e}_n) \\ &= \begin{bmatrix} L(\vec{e}_1) & L(\vec{e}_2) & \cdots & L(\vec{e}_n) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \end{aligned}$$

So, just as before, if we want to find the standard matrix $[L]$ for a linear mapping $L$, we see that

$$[L] = \begin{bmatrix} L(\vec{e}_1) & L(\vec{e}_2) & \cdots & L(\vec{e}_n) \end{bmatrix}$$

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### Example

Find the standard matrix for the linear mapping $L : \mathbb{C}^3 \to \mathbb{C}^3$ defined by

$$L(z_1, z_2, z_3) = (z_1 + (1-3i)z_3, 4iz_2, 2z_1 + (1-5i)z_2)$$

First, we compute the following:

$L(\vec{e}_1) = L(1,0,0) = (1 + (1-3i)(0), 4i(0), 2(1) + (1-5i)(0)) = (1, 0, 2)$

$L(\vec{e}_2) = L(0,1,0) = (0 + (1-3i)(0), 4i(1), 2(0) + (1-5i)(1)) = (0, 4i, 1-5i)$

$L(\vec{e}_3) = L(0,0,1) = (0 + (1-3i)(1), 4i(0), 2(0) + (1-5i)(0)) = (1-3i, 0, 0)$

And this means that $[L] = \begin{bmatrix} 1 & 0 & 1-3i \\ 0 & 4i & 0 \\ 2 & 1-5i & 0 \end{bmatrix}$.