The Principal Axis Theorem

The goal of this lecture is to show that every symmetric matrix is orthogonally diagonalizable. We spent time in the last lecture looking at the process of finding an orthogonal matrix P to diagonalize a symmetric matrix A. During the process, we can guarantee that we can find an orthonormal basis for each of the eigenspaces for A, but in the end we always found that making a matrix P from these orthonormal basis vectors created an orthogonal matrix.

Now, since all of the vectors from our orthonormal bases would have norm 1, this feature of the columns of P is not surprising. It is also obvious that the columns of P from the same eigenspace would be orthogonal, as they are part of the same orthonormal basis. The thing that separates our situation from any random diagonalization process is that the vectors from one eigenspace are orthogonal to the vectors from another eigenspace. In order to prove this crucial fact, we first want to show the following.

The Principal Axis Theorem

Lemma 8.1.1

An $n \times n$ matrix A is symmetric if and only if

$$\vec{x} \cdot (A\vec{y}) = (A\vec{x}) \cdot \vec{y}$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

Proof

We need to prove both directions of our "if and only if" statement.

($\Rightarrow$): If A is symmetric, then $\vec{x} \cdot (A\vec{y}) = (A\vec{x}) \cdot \vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

Suppose A is symmetric, and let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then we can rewrite the dot product $\vec{x} \cdot (A\vec{y})$ as the matrix product $\vec{x}^T A\vec{y}$.

Next, we use the fact that $A^T = A$ to get that

$$\vec{x} \cdot (A\vec{y}) = \vec{x}^T A^T \vec{y}$$

Regrouping, we see that

$$\vec{x}^T A^T \vec{y} = (A\vec{x})^T \vec{y}$$

which we can write as the dot product $(A\vec{x}) \cdot \vec{y}$.

So we see that

$$\vec{x} \cdot (A\vec{y}) = (A\vec{x}) \cdot \vec{y}$$

as desired.

The Principal Axis Theorem

Proof

($\Leftarrow$): If $\vec{x} \cdot (A\vec{y}) = (A\vec{x}) \cdot \vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, then A is symmetric.

Suppose that $\vec{x} \cdot (A\vec{y}) = (A\vec{x}) \cdot \vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. As before, we can rewrite these dot products as matrix products, getting that

$$\vec{x}^T A \vec{y} = (A\vec{x})^T \vec{y} \text{ for all } \vec{x}, \vec{y} \in \mathbb{R}^n$$

If we set

$$B = \vec{x}^T A \text{ and } C = (A\vec{x})^T$$

then we have that

$$B\vec{y} = C\vec{y} \text{ for all } \vec{y} \in \mathbb{R}^n$$

so by Theorem 3.1.4, $B = C$. That is, $\vec{x}^T A = (A\vec{x})^T$.

Taking the transpose of both sides, we get that

$$(\vec{x}^T A)^T = ((A\vec{x})^T)^T = A\vec{x}$$
$$A\vec{x} = (\vec{x}^T A)^T = A^T (\vec{x}^T)^T = A^T \vec{x}$$

so

And since

$$A\vec{x} = A^T \vec{x} \text{ for all } \vec{x} \in \mathbb{R}^n$$

by another use of Theorem 3.1.4, we see that $A = A^T$, and thus that A is symmetric.

The Principal Axis Theorem

Theorem 8.1.2

If $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of a symmetric matrix A corresponding to distinct eigenvalues λ_1 and λ_2, then $\vec{v}_1$ is orthogonal to $\vec{v}_2$.

Proof

By the definition of eigenvalues and eigenvectors, we have that

$$A\vec{v}_1 = \lambda_1 \vec{v}_1 \text{ and } A\vec{v}_2 = \lambda_2 \vec{v}_2$$

We want to show that $\vec{v}_1 \cdot \vec{v}_2 = 0$. In a leap of creativity, what we are actually going to show is that

$$(\lambda_1 - \lambda_2)(\vec{v}_1 \cdot \vec{v}_2) = 0$$

This is equivalent, since $\lambda_1 \neq \lambda_2$ means that $(\lambda_1 - \lambda_2)(\vec{v}_1 \cdot \vec{v}_2) = 0$ if and only if $\vec{v}_1 \cdot \vec{v}_2 = 0$.

So, to show that $(\lambda_1 - \lambda_2)(\vec{v}_1 \cdot \vec{v}_2) = 0$, we first note that

$$\lambda_1(\vec{v}_1 \cdot \vec{v}_2) = (\lambda_1 \vec{v}_1) \cdot \vec{v}_2 = (A\vec{v}_1) \cdot \vec{v}_2$$

We also know that

$$\lambda_2(\vec{v}_1 \cdot \vec{v}_2) = \vec{v}_1 \cdot (\lambda_2 \vec{v}_2) = \vec{v}_1 \cdot (A\vec{v}_2)$$

From Lemma 8.1.1, we know that $(A\vec{v}_1) \cdot \vec{v}_2 = \vec{v}_1 \cdot (A\vec{v}_2)$. So we see that

$$\lambda_1(\vec{v}_1 \cdot \vec{v}_2) = \lambda_2(\vec{v}_1 \cdot \vec{v}_2)$$

which means that

$$\lambda_1(\vec{v}_1 \cdot \vec{v}_2) - \lambda_2(\vec{v}_1 \cdot \vec{v}_2) = (\lambda_1 - \lambda_2)(\vec{v}_1 \cdot \vec{v}_2) = 0$$

as desired.

The Principal Axis Theorem

Theorem 8.1.3

If A is symmetric, then all of the eigenvalues of A are real.

Lemma 8.1.4

Suppose that λ_1 is an eigenvalue of the $n \times n$ symmetric matrix A, with corresponding unit eigenvector $\vec{v}_1$. Then there is an orthogonal matrix P whose first column is $\vec{v}_1$, such that

$$P^T A P = \begin{bmatrix} \lambda_1 & O_{1,n-1} \\ O_{n-1,1} & A_1 \end{bmatrix}$$

where A_1 is an $(n-1) \times (n-1)$ symmetric matrix and $O_{m,n}$ is the $m \times n$ zero matrix. (That is, $P^T A P$ is a matrix whose first entry is λ_1, the remaining entries in the first row and first column are all zeroes, and the $(n-1) \times (n-1)$ submatrix we get by removing the first row and first column is a symmetric matrix.)

Proof

First, we know that we can extend the set $\{\vec{v}_1\}$ to a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$.

And then we can use the Gram-Schmidt procedure on this basis to get an orthonormal basis $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ for $\mathbb{R}^n$. But, since $\vec{v}_1$ is already a unit vector, the Gram-Schmidt procedure will set $\vec{w}_1 = \vec{v}_1$, so $\mathcal{B} = \{\vec{v}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ is an orthonormal basis for $\mathbb{R}^n$. Let

$$P = [\vec{v}_1 \quad \vec{w}_2 \quad \cdots \quad \vec{w}_n]$$

The Principal Axis Theorem

Proof

First, we note that $P^T A P$ is symmetric, since

$$(P^T A P)^T = P^T (P^T A)^T = P^T A (P^T)^T = P^T A P$$

But what is $P^T A P$? Multiplying it out, we get:

$$P^T A P = \begin{bmatrix} \vec{v}_1^T \\ \vec{w}_2^T \\ \vdots \\ \vec{w}_n^T \end{bmatrix} A [\vec{v}_1 \quad \vec{w}_2 \quad \cdots \quad \vec{w}_n] = \begin{bmatrix} \vec{v}_1^T \\ \vec{w}_2^T \\ \vdots \\ \vec{w}_n^T \end{bmatrix} [A\vec{v}_1 \quad A\vec{w}_2 \quad \cdots \quad A\vec{w}_n]$$

$$= \begin{bmatrix} \vec{v}_1^T A\vec{v}_1 & \vec{v}_1^T A\vec{w}_2 & \cdots & \vec{v}_1^T A\vec{w}_n \\ \vec{w}_2^T A\vec{v}_1 & \vec{w}_2^T A\vec{w}_2 & \cdots & \vec{w}_2^T A\vec{w}_n \\ \vdots & \vdots & & \vdots \\ \vec{w}_n^T A\vec{v}_1 & \vec{w}_n^T A\vec{w}_2 & \cdots & \vec{w}_n^T A\vec{w}_n \end{bmatrix}$$

$$= \begin{bmatrix} \vec{v}_1 \cdot A\vec{v}_1 & \vec{v}_1 \cdot A\vec{w}_2 & \cdots & \vec{v}_1 \cdot A\vec{w}_n \\ \vec{w}_2 \cdot A\vec{v}_1 & \vec{w}_2 \cdot A\vec{w}_2 & \cdots & \vec{w}_2 \cdot A\vec{w}_n \\ \vdots & \vdots & & \vdots \\ \vec{w}_n \cdot A\vec{v}_1 & \vec{w}_n \cdot A\vec{w}_2 & \cdots & \vec{w}_n \cdot A\vec{w}_n \end{bmatrix}$$

The Principal Axis Theorem

Proof

Since $\vec{v}_1$ is a unit eigenvector of λ_1, we have that

$$\vec{v}_1 \cdot A\vec{v}_1 = \vec{v}_1 \cdot (\lambda_1 \vec{v}_1) = \lambda_1 (\vec{v}_1 \cdot \vec{v}_1) = \lambda_1$$

So we've shown that $(P^T A P)_{11} = \lambda_1$.

What about the rest of the first column?

Well, since B is orthonormal, we get that

$$\vec{w}_i \cdot A\vec{v}_1 = \vec{w}_i \cdot (\lambda_1 \vec{v}_1) = \lambda_1 (\vec{w}_i \cdot \vec{v}_1) = 0$$

So $(P^T A P)_{i1} = 0$ for all $2 \leq i \leq n$. And the fact that $(P^T A P)$ is symmetric tells us that $(P^T A P)_{1i} = 0$ for all $2 \leq i \leq n$. And this means that

$$P^T A P = \begin{bmatrix} \lambda_1 & O_{1,n-1} \\ O_{n-1,1} & A_1 \end{bmatrix}$$

where A_1 must be symmetric because $P^T A P$ is symmetric.

The Principal Axis Theorem

Theorem 8.1.5 (The Principal Axis Theorem)

Suppose A is an $n \times n$ symmetric matrix. Then there exists an orthogonal matrix P and a diagonal matrix D such that $P^T A P = D$. That is, every symmetric matrix is orthogonally diagonalizable.

The Principal Axis Theorem

Proof

The proof is by induction on n, the size of our symmetric matrix A. The base case is when $n = 1$, which means $A = [a]$, and A is diagonalized by the orthogonal matrix

$$P = [1] \text{ to } P^T A P = [1][a][1] = [a]$$

For our induction step, suppose we know that all $(n-1) \times (n-1)$ symmetric matrices are orthogonally diagonalizable, and let A be an $n \times n$ symmetric matrix.

By Theorem 8.1.3, we know that A has a real eigenvalue λ_1, and let $\vec{v}_1$ be a unit eigenvector corresponding to λ_1.

Then by Lemma 8.1.4 we know that there is an orthogonal matrix $R = \begin{bmatrix} \vec{v}_1 & \vec{w}_2 & \dots & \vec{w}_n \end{bmatrix}$ such that

$$R^T A R = \begin{bmatrix} \lambda_1 & O_{1,n-1} \\ O_{n-1,1} & A_1 \end{bmatrix}$$

where A_1 is an $(n-1) \times (n-1)$ symmetric matrix.

By our inductive hypothesis, there is an $(n-1) \times (n-1)$ orthogonal matrix P_1 and an $(n-1) \times (n-1)$ diagonal matrix D_1 such that $P_1^T A_1 P_1 = D_1$. Define the $n \times n$ matrix P_2 as follows:

$$P_2 = \begin{bmatrix} 1 & O_{1,n-1} \\ O_{n-1,1} & P_1 \end{bmatrix}$$

Note that the first column of P_2 is the vector $\vec{e}_1$, so it is a unit vector.

Moreover, all the other columns of P_2 are also unit vectors, as their norm is not changed from P_1 by adding a 0 as the first entry.

The Principal Axis Theorem

Proof

Moreover, all the columns of P_2 are orthogonal, either inherited from P_1, or because they have no non-zero entries in common with the first column. As such P_2 is also an orthogonal matrix.

Since the product of orthogonal matrices is orthogonal, the matrix $P = RP_2$ is an $n \times n$ orthogonal matrix such that

$$
\begin{aligned}
P^T A P &= (RP_2)^T A (RP_2) \\
&= P_2^T (R^T A R) P_2 \\
&= P_2^T \begin{bmatrix} \lambda_1 & O_{1,n-1} \\ O_{n-1,1} & A_1 \end{bmatrix} P_2 \\
&= \begin{bmatrix} 1 & O_{1,n-1} \\ O_{n-1,1} & P_1^T \end{bmatrix} \begin{bmatrix} \lambda_1 & O_{1,n-1} \\ O_{n-1,1} & A_1 \end{bmatrix} \begin{bmatrix} 1 & O_{1,n-1} \\ O_{n-1,1} & P_1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & O_{1,n-1} \\ O_{n-1,1} & P_1^T \end{bmatrix} \begin{bmatrix} \lambda_1 & O_{1,n-1} \\ O_{n-1,1} & A_1 P_1 \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 & O_{1,n-1} \\ O_{n-1,1} & P_1^T A_1 P_1 \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 & O_{1,n-1} \\ O_{n-1,1} & D_1 \end{bmatrix}
\end{aligned}
$$

where $\begin{bmatrix} \lambda_1 & O_{1,n-1} \\ O_{n-1,1} & D_1 \end{bmatrix}$ is our diagonal matrix D.