Sketching Quadratic Forms

Let $Q(\vec{x}) = ax_1^2 + bx_1x_2 + cx_2^2$ be a quadratic form in $\mathbb{R}^2$.
In many applications of quadratic forms it is important to be able to sketch the graph of a quadratic form $Q(\vec{x}) = k$ for some $k \in \mathbb{R}$.
In general, it is not easy to graph $ax_1^2 + bx_1x_2 + cx_2^2 = k$. How can we make this easier to graph?

There exists an orthogonal matrix P such that the change of variables $\vec{y} = P^T\vec{x}$ will give us

$$k = Q(\vec{x}) = \lambda_1 y_1^2 + \lambda_2 y_2^2$$

Observe that this is now quite easy to draw as these are just conic sections.
The possibilities are demonstrated in the following table:

	$k > 0$	$k = 0$	$k < 0$
$\lambda_1 > 0, \lambda_2 > 0$	ellipse	point$(0,0)$	DNE
$\lambda_1 < 0, \lambda_2 < 0$	DNE	point$(0,0)$	ellipse
$\lambda_1\lambda_2 < 0$	hyperbola	asymptotes for hyperbola/intersecting lines	hyperbola
$\lambda_1 = 0, \lambda_2 > 0$	parallel lines	line $y_2 = 0$	DNE
$\lambda_1 = 0, \lambda_2 < 0$	DNE	line $y_2 = 0$	parallel lines

Note: DNE – Does Not Exist.

Sketching Quadratic Forms

Theorem 10.4.1

If $Q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$ where a, b, c are not all zero, then there exists an orthogonal matrix P, which corresponds to a rotation, such that the change of variables $\vec{y} = P^T\vec{x}$ brings $Q(\vec{x})$ into diagonal form.

Proof

Let A be the symmetric matrix corresponding to $Q(x_1, x_2)$, and $\vec{v}_1 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ be a unit eigenvector of A.

Since $\vec{v}_1$ is a unit vector, we have that

$$1 = \|\vec{v}_1\|^2 = a_1^2 + a_2^2$$

Therefore, we see that the components of $\vec{v}_1$ lie on the unit circle.
That is, there exists θ such that $a_1 = \cos\theta$ and $a_2 = \sin\theta$.
By the Principal Axis Theorem, there must exist an orthogonal basis of eigenvectors of A.
If we let $\vec{v}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$, then $\vec{v}_2$ is orthogonal to $\vec{v}_1$ and hence must be an eigenvector of A.
Therefore, the matrix

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is an orthogonal matrix which diagonalizes A.
Observe that P is a rotation matrix as required.

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Sketching Quadratic Forms

Example

Graph $3x_1^2 + 4x_1x_2 + 3x_2^2 = 5$.

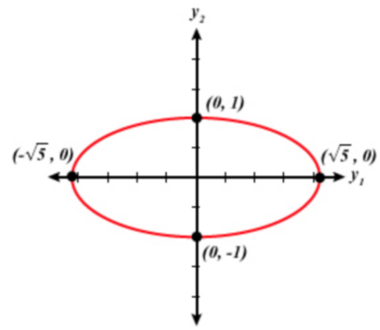
Solution

We have corresponding symmetric matrix $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$.

We find that the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 5$.

Theorem 10.4.1 tells us that the graph of $3x_1^2 + 4x_1x_2 + 3x_2^2 = 5$ is going to be a rotation of the graph of
$5 = \lambda_1 y_1^2 + \lambda_2 y_2^2 = y_1^2 + 5y_2^2$.

Since A has both positive eigenvalues, A is positive definite, and so from the table we did, we have that the graph of this is an ellipse.



Let's first sketch $y_1^2 + 5y_2^2 = 5$ in the y_1y_2-plane.
We can easily sketch this ellipse by determining the y_1 and y_2 intercepts.

However, we want to find the graph of $3x_1^2 + 4x_1x_2 + 3x_2^2 = 5$ in the x_1x_2-plane.

Sketching Quadratic Forms

Example

Graph $3x_1^2 + 4x_1x_2 + 3x_2^2 = 5$.

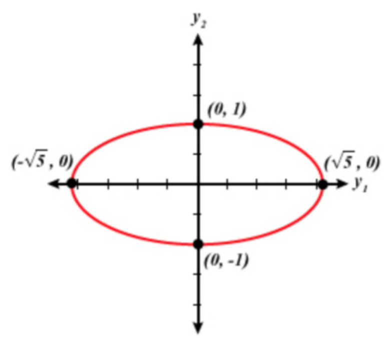
Solution

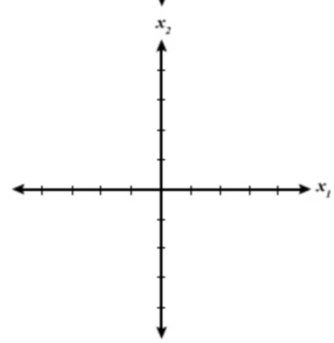
To get the diagonal form $y_1^2 + 5y_2^2 = 5$, we needed to use a change of variables $\vec{y} = P^T\vec{x}$.
To change the graph from the y_1y_2-plane to the x_1x_2-plane, we can use the change of variables $\vec{x} = P\vec{y}$.
We need to find a matrix P that orthogonally diagonalizes A.

We find that a unit eigenvector for $\lambda_1 = 1$ is $\vec{v}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, and a unit eigenvector for $\lambda_2 = 5$ is $\vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

Hence, $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

Notice that P is not a rotation matrix!
The statement of Theorem 10.4.1 only says that there exists a rotation, it does not say that it has to be a rotation.
This choice of P gives a rotation and a reflection.





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Sketching Quadratic Forms

Example

Graph $3x_1^2 + 4x_1x_2 + 3x_2^2 = 5$.

Solution

Using the change of variables $\vec{x} = P\vec{y}$, we can transform any vector in the y_1y_2-plane to a vector in the x_1x_2-plane.

Notice that the y_1-axis is just the line spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Thus, using the change of variables $\vec{x} = P\vec{y}$ we get

$$\vec{x} = [\,\vec{v}_1 \quad \vec{v}_2\,] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{v}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

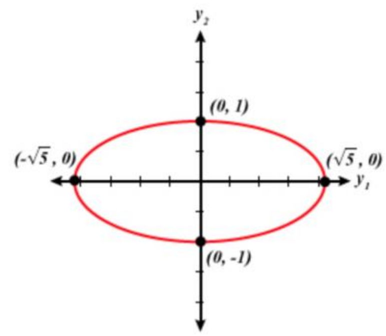
Thus, the y_1-axis in the x_1x_2-plane is the line spanned by $\vec{v}_1$.

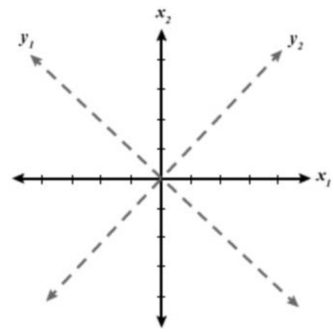
Similarly, the y_2-axis is spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Therefore, in the x_1x_2-plane we have the y_2 axis is the line spanned by

$$\vec{x} = [\,\vec{v}_1 \quad \vec{v}_2\,] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

We now just copy our ellipse from the graph in the y_1y_2-plane onto the y_1y_2-axes in the x_1x_2-plane to get the sketch of the graph in the x_1x_2-plane.

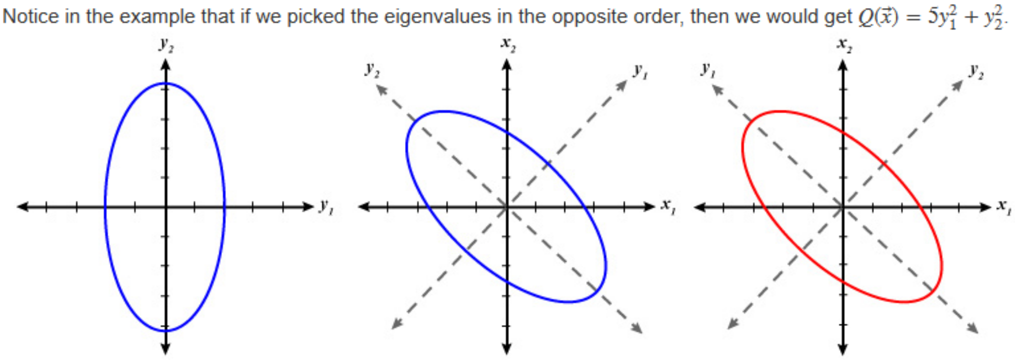




Sketching Quadratic Forms

We know in diagonalization that we can pick eigenvalues in any order as long as the columns in P correspond to the order of the eigenvalues in $P^TAP = D$.

Notice in the example that if we picked the eigenvalues in the opposite order, then we would get $Q(\vec{x}) = 5y_1^2 + y_2^2$.



However, we would also have the eigenvectors in the opposite order as well.

In particular, we would have the y_1 axis in the x_1x_2-plane would be in the direction of the eigenvector $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ corresponding to $\lambda_2 = 5$, and the y_2 axis in the x_1x_2-plane would be in the direction of the eigenvector $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ corresponding to $\lambda_1 = 1$.

Thus, we would get the picture, which, of course, matches our answer in the example.

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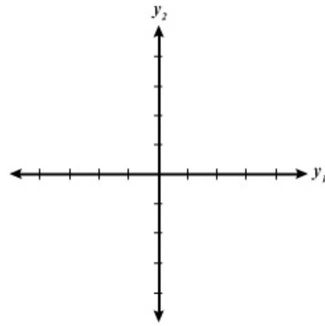
Sketching Quadratic Forms

Example

Sketch $3x_1^2 + 4x_1x_2 = 16$.

Solution



We have $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$.

We find the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -1$.

Therefore, we have the quadratic form $4y_1^2 - y_2^2$.

Hence, we see that the graph of $4y_1^2 - y_2^2 = 16$ is a hyperbola.

Hence, the graph of $3x_1^2 + 4x_1x_2 = 16$ is going to be a rotated hyperbola.

We find corresponding eigenvectors of A are $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for $\lambda_1 = 4$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ for $\lambda_2 = -1$.

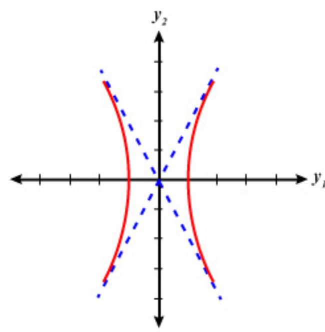
Thus, an orthogonal matrix P which diagonalizes A is $P = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$.

Sketching Quadratic Forms

Example

Sketch $3x_1^2 + 4x_1x_2 = 16$.

Solution



To accurately sketch the hyperbola $4y_1^2 - y_2^2 = 16$ we need to find the asymptotes of the hyperbola.

To do this, we set the right hand side to 0 to get

$$\begin{aligned} 4y_1^2 - y_2^2 &= 0 \\ 4y_1^2 &= y_2^2 \\ \pm 2y_1 &= y_2 \end{aligned}$$

For the asymptote $2y_1 = y_2$, if take $y_1 = 1$, we get $y_2 = 2$, so a direction vector for this asymptote is $\vec{w}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Similarly, for $-2y_1 = y_2$, if we take $y_1 = 1$, we get $y_2 = -2$, so a direction vector for this asymptote is $\vec{w}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Notice that taking $y_1 = 0$ in $4y_1^2 - y_2^2 = 16$ gives $-y_2^2 = 16$ which is impossible.

On the other hand, if we take $y_2 = 0$, we get $y_1 = \pm 2$.

We now want to use this to sketch $3x_1^2 + 4x_1x_2 = 16$ in the x_1x_2-plane.

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Sketching Quadratic Forms

Example

Sketch $3x_1^2 + 4x_1x_2 = 16$.

Solution

We saw in the last example that we can use the change of variables $\vec{x} = P\vec{y}$ to get that the y_1 and y_2-axes in the x_1x_2-plane are in the direction of the eigenvectors $\vec{v}_1$ and $\vec{v}_2$.

We can again use the change of variables to transfer the equations of the asymptotes from the y_1y_2-plane into the x_1x_2-plane.

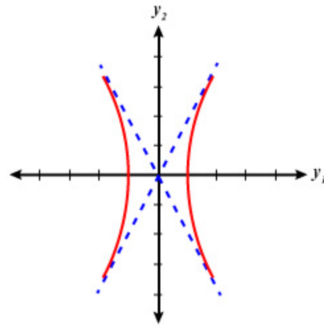
For the asymptote $2y_1 = y_2$ we use the direction vector $\vec{w}_1$ to get

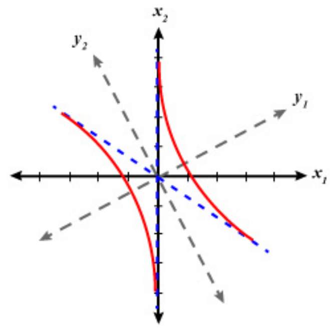
$$\vec{x}_1 = P\vec{w}_1 = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{5} \end{bmatrix}$$

For the asymptote $-2y_1 = y_2$ we use $\vec{w}_2$ to get

$$\vec{x}_2 = P\vec{w}_2 = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{5} \\ -3/\sqrt{5} \end{bmatrix}$$

We can transcribe our hyperbola from the y_1y_2-plane onto this.
In particular, we saw the graph open in the y_1-direction.





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