Change of Coordinates

Last Lecture

- We looked at how to find the coordinate vector of a vector with respect to a basis.

In This Lecture

- We will find a quick way of converting the coordinates of a vector with respect to one basis to the coordinates of the vector with respect to another basis.

Change of Coordinates

Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\mathcal{C} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ be two bases for $\mathbb{R}^3$.
Our goal is to find the $\mathcal{C}$-coordinate vector of $\vec{x}$ in $\mathbb{R}^3$ if we are only given the $\mathcal{B}$-coordinate vector of $\vec{x}$.
Assume that $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.
That means that $\vec{x} = b_1\vec{v}_1 + b_2\vec{v}_2 + b_3\vec{v}_3$.
Observe that

$$[\vec{x}]_{\mathcal{C}} = \left[b_1\vec{v}_1 + b_2\vec{v}_2 + b_3\vec{v}_3\right]_{\mathcal{C}} = b_1[\vec{v}_1]_{\mathcal{C}} + b_2[\vec{v}_2]_{\mathcal{C}} + b_3[\vec{v}_3]_{\mathcal{C}} = \left[[\vec{v}_1]_{\mathcal{C}} \quad [\vec{v}_2]_{\mathcal{C}} \quad [\vec{v}_3]_{\mathcal{C}}\right]\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

We call this matrix ${}_{\mathcal{C}}P_{\mathcal{B}} = \left[[\vec{v}_1]_{\mathcal{C}} \quad [\vec{v}_2]_{\mathcal{C}} \quad [\vec{v}_3]_{\mathcal{C}}\right]$ the change of coordinates matrix from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates.

Change of Coordinates

Example

Let $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^3$. Find the $\mathcal{C}$-coordinate vector of any $\vec{x} \in \mathbb{R}^3$.

Solution

To find the $\mathcal{C}$-coordinates of any vector $\vec{x}$ in $\mathbb{R}^3$, we will find the change of coordinates matrix from $\mathcal{C}$-coordinates to standard coordinates.

Our previous work shows us that

$$_{\mathcal{C}}P_{\mathcal{B}} = \begin{bmatrix} [\vec{e}_1]_{\mathcal{C}} & [\vec{e}_2]_{\mathcal{C}} & [\vec{e}_3]_{\mathcal{C}} \end{bmatrix}$$

We need to find $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ such that

$$a_1 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$b_1 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + b_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + b_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Change of Coordinates

Example

Let $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^3$. Find the $\mathcal{C}$-coordinate vector of any $\vec{x} \in \mathbb{R}^3$.

Solution

Row reducing the corresponding multiple augmented system gives

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 4 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/5 & -1/5 & -1 \\ 0 & 1 & 0 & 7/5 & -4/5 & -1 \\ 0 & 0 & 1 & -4/5 & 3/5 & 1 \end{array}\right]$$

Thus,

$$_{\mathcal{C}}P_{\mathcal{B}} = \begin{bmatrix} [\vec{e}_1]_{\mathcal{C}} & [\vec{e}_2]_{\mathcal{C}} & [\vec{e}_3]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 3/5 & -1/5 & -1 \\ 7/5 & -4/5 & -1 \\ -4/5 & 3/5 & 1 \end{bmatrix}$$

So,

$$[\vec{x}]_{\mathcal{C}} = {}_{\mathcal{C}}P_{\mathcal{B}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{5}x_1 - \frac{1}{5}x_2 - x_3 \\ \frac{7}{5}x_1 - \frac{4}{5}x_2 - x_3 \\ -\frac{4}{5}x_1 + \frac{3}{5}x_2 + x_3 \end{bmatrix}$$

Change of Coordinates

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $\mathcal{C}$ both be bases for a vector space $\mathbb{V}$.
If $\vec{x} = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n$, then

$$[\vec{x}]_\mathcal{C} = [b_1\vec{v}_1 + \cdots + b_n\vec{v}_n]_\mathcal{C} = b_1[\vec{v}_1]_\mathcal{C} + \cdots + b_n[\vec{v}_n]_\mathcal{C} = \begin{bmatrix} [\vec{v}_1]_\mathcal{C} & \cdots & [\vec{v}_n]_\mathcal{C} \end{bmatrix}[\vec{x}]_\mathcal{B}$$

Definition: Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $\mathcal{C}$ both be bases for a vector space $\mathbb{V}$. The matrix

$${}_\mathcal{C}P_\mathcal{B} = \begin{bmatrix} [\vec{v}_1]_\mathcal{C} & \cdots & [\vec{v}_n]_\mathcal{C} \end{bmatrix}$$

is called the change of coordinates matrix from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates. It satisfies

$$[\vec{x}]_\mathcal{C} = {}_\mathcal{C}P_\mathcal{B}[\vec{x}]_\mathcal{B}$$

Change of Coordinates

Example

Let $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1, x+1, (x+1)^2\}$. Find the change of coordinates matrix ${}_\mathcal{C}P_\mathcal{B}$ from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates, and the change of coordinates matrix ${}_\mathcal{B}P_\mathcal{C}$ from $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates.

Solution

To find ${}_\mathcal{C}P_\mathcal{B}$ we need to calculate the $\mathcal{C}$-coordinate vectors of the vectors in $\mathcal{B}$.
That is, we need to solve

$$\begin{aligned} 1 &= a_1(1) + a_2(x+1) + a_3(x+1)^2 = (a_1 + a_2 + a_3)(1) + (a_2 + 2a_3)x + a_3x^2 \\ x &= b_1(1) + b_2(x+1) + b_3(x+1)^2 = (b_1 + b_2 + b_3)(1) + (b_2 + 2b_3)x + b_3x^2 \\ x^2 &= c_1(1) + c_2(x+1) + c_3(x+1)^2 = (c_1 + c_2 + c_3)(1) + (c_2 + 2c_3)x + c_3x^2 \end{aligned}$$

Row reducing the corresponding multiple augmented matrix gives

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Thus,

$${}_\mathcal{C}P_\mathcal{B} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Change of Coordinates

Example

Let $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1, x+1, (x+1)^2\}$. Find the change of coordinates matrix ${}_{\mathcal{C}}P_{\mathcal{B}}$ from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates, and the change of coordinates matrix ${}_{\mathcal{B}}P_{\mathcal{C}}$ from $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates.

Solution

For ${}_{\mathcal{B}}P_{\mathcal{C}}$, we need to find the $\mathcal{B}$-coordinate vectors of the vectors in $\mathcal{C}$.
Since $\mathcal{B}$ is the standard basis, we get these by inspection.
In particular, we have

$$[1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [1+x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad [(1+x)^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Thus,

$${}_{\mathcal{B}}P_{\mathcal{C}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that we can check our answer in this example by using ${}_{\mathcal{C}}P_{\mathcal{B}}$ to convert $p(x) = a + bx + cx^2$ from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates and then using ${}_{\mathcal{B}}P_{\mathcal{C}}$ to convert it back.

Change of Coordinates

Example

Let $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1, x+1, (x+1)^2\}$. Find the change of coordinates matrix ${}_{\mathcal{C}}P_{\mathcal{B}}$ from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates, and the change of coordinates matrix ${}_{\mathcal{B}}P_{\mathcal{C}}$ from $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates.

Solution

We have $[p(x)]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Thus,

$$[p(x)]_{\mathcal{C}} = {}_{\mathcal{C}}P_{\mathcal{B}}[p(x)]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a-b+c \\ b-2c \\ c \end{bmatrix}$$

Then,

$$\begin{aligned} [p(x)]_{\mathcal{B}} &= {}_{\mathcal{B}}P_{\mathcal{C}}[p(x)]_{\mathcal{C}} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a-b+c \\ b-2c \\ c \end{bmatrix} \\ &= \begin{bmatrix} (a-b+c)+(b-2c)+c \\ (b-2c)+2c \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \end{aligned}$$

as required.

Change of Coordinates

Let $A = {}_{\mathcal{B}}P_{\mathcal{C}}\, {}_{\mathcal{C}}P_{\mathcal{B}}$.
We get that

$$\begin{aligned} A[\vec{x}]_{\mathcal{B}} &= {}_{\mathcal{B}}P_{\mathcal{C}}\, {}_{\mathcal{C}}P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} \\ &= {}_{\mathcal{B}}P_{\mathcal{C}}[\vec{x}]_{\mathcal{C}} \\ &= [\vec{x}]_{\mathcal{B}} \end{aligned}$$

Using Theorem 3.1.4, this implies that $A = I$.

Theorem 4.3.3

If $\mathcal{B}$ and $\mathcal{C}$ are both bases of a finite dimensional vector space $\mathbb{V}$, then the change of coordinate matrices ${}_{\mathcal{C}}P_{\mathcal{B}}$ and ${}_{\mathcal{B}}P_{\mathcal{C}}$ satisfy

$${}_{\mathcal{C}}P_{\mathcal{B}}\, {}_{\mathcal{B}}P_{\mathcal{C}} = I = {}_{\mathcal{B}}P_{\mathcal{C}}\, {}_{\mathcal{C}}P_{\mathcal{B}}$$

A complete proof is provided in the course notes.