The Cofactor Method and Cramer's Rule

Previously

- We proved a matrix is invertible if and only if the determinant of the matrix is 0.
- We saw that the determinant of a matrix is calculated in terms of cofactors of the matrix.

In This Lecture

- We will learn how to calculate the inverse of a matrix using the cofactors of the matrix.
- We will apply this to get another way to solve a system of linear equations.

The Cofactor Method

Definition: Let A be an $n \times n$ matrix. The adjugate of A is the matrix defined by

$$\left(\operatorname{adj} A\right)_{ij} = C_{ji}$$

In particular, $\operatorname{adj} A = (\operatorname{cof} A)^T$.

Example

Calculate the adjugate of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Solution

We have

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem 5.4.2

If A is an invertible $n \times n$ matrix, then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

This is called the Cofactor Method of finding the inverse.

The Cofactor Method

Example

Find the inverse of $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix}$ using the cofactor method.

Solution

We first find

$$\det A = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \\ 4 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ -7 & 3 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 13$$

We get

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ -4 & 6 & 1 \\ -7 & 4 & 5 \end{bmatrix}$$

Hence,

$$A^{-1} = \frac{1}{13}\begin{bmatrix} 5 & -1 & 2 \\ -4 & 6 & 1 \\ -7 & 4 & 5 \end{bmatrix}$$

The Cofactor Method

There are two reasons why we would use the cofactor method.

First, the cofactor method gives us a formula for each entry of the inverse of a matrix.

In particular, the ij-th entry of A^{-1} is $\frac{1}{\det A} C_{ji}$.

Example

Let $A = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix}$. Assuming that A is invertible, find A^{-1}.

Solution

We get

$$\det A = c(ae - bd) = ace - bcd$$

Assuming $\det A \neq 0$, we then find that

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{ace - bcd}\begin{bmatrix} ce & 0 & -bc \\ 0 & ae - bd & 0 \\ -cd & 0 & ac \end{bmatrix}$$

Cramer's Rule

Consider the system of linear equations $A\vec{x} = \vec{b}$ where A is an $n \times n$ matrix.
If A is invertible, then we know this system has a unique solution $\vec{x} = A^{-1}\vec{b}$.
But, we have just seen that

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

So, we may write the solution as

$$\vec{x} = \frac{1}{\det A} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Thus, the i-th component of $\vec{x}$ in the solution of $A\vec{x} = \vec{b}$ is

$$x_i = \frac{b_1 C_{1i} + \cdots + b_n C_{ni}}{\det A}$$

If we let

$$A_i = \begin{bmatrix} a_{11} & \cdots & a_{1(i-1)} & b_1 & a_{1(i+1)} & \cdots & a_{1n} \\ \vdots & & & \vdots & & & \vdots \\ a_{n1} & \cdots & a_{n(i-1)} & b_n & a_{n(i+1)} & \cdots & a_{nn} \end{bmatrix}$$

then we get $x_i = \dfrac{\det A_i}{\det A}$.

Cramer's Rule

Theorem 5.4.3 - Cramer's Rule

If A is an $n \times n$ invertible matrix, then the solution $\vec{x}$ of $A\vec{x} = \vec{b}$ is given by

$$x_i = \frac{\det A_i}{\det A}, \quad 1 \le i \le n$$

where A_i is the matrix obtained from A by replacing the i-th column of A by $\vec{b}$.

Cramer's Rule

Example 1

Solve the system of linear equations

$$\begin{aligned} 5x_1 + x_2 - x_3 &= 4 \\ 9x_1 + x_2 - x_3 &= 1 \\ x_1 - x_2 + 5x_3 &= 2 \end{aligned}$$

Solution

The coefficient matrix is $A = \begin{bmatrix} 5 & 1 & -1 \\ 9 & 1 & -1 \\ 1 & -1 & 5 \end{bmatrix}$, so $\det A = -16$.

Hence, Cramer's Rule gives

$$x_1 = \frac{1}{-16} \begin{vmatrix} 4 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & -1 & 5 \end{vmatrix} = -\frac{3}{4}$$

$$x_2 = \frac{1}{-16} \begin{vmatrix} 5 & 4 & -1 \\ 9 & 1 & -1 \\ 1 & 2 & 5 \end{vmatrix} = \frac{83}{8}$$

$$x_3 = \frac{1}{-16} \begin{vmatrix} 5 & 1 & 4 \\ 9 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \frac{21}{8}$$

Cramer's Rule

Example 2

Assuming that $A = \begin{bmatrix} a & 1 & 4 \\ 0 & b & 3 \\ c & 1 & -1 \end{bmatrix}$ is invertible, find x_1 in the solution of

$$A\vec{x} = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

Solution

First, we have

$$\det A = -ab - 3a + 3c - 4cb$$

Thus, Cramer's Rule gives

$$x_1 = \frac{1}{\det A} \begin{vmatrix} -2 & 1 & 4 \\ -3 & b & 3 \\ 1 & 1 & -1 \end{vmatrix} = \frac{-2b - 6}{-ab - 3a + 3c - 4cb}$$