Determinants and Row Operations

Last Lecture

- We saw that it is easy to calculate the determinant of a triangular matrix.

In This Lecture

- We explore performing elementary row operations to get a triangular matrix.

Determinants and Row Operations

Theorem 5.3.3

If B is the matrix obtained from A by multiplying one row of A by a non-zero constant c, then $\det B = c \det A$.

Theorem 5.3.4

If B is the matrix obtained from A by swapping two rows of A, then $\det B = -\det A$.

Theorem 5.3.5

If B is the matrix obtained from A by adding c times the k-th row of A to the j-th row, then $\det B = \det A$.

Determinants and Row Operations

Example 1

Calculate $\begin{vmatrix} 2 & 1 & 3 \\ 4 & 3 & 8 \\ -2 & 3 & -2 \end{vmatrix}$.

Solution

Since adding a multiple of one row to another does not change the determinant

$$\begin{aligned} \begin{vmatrix} 2 & 1 & 3 \\ 4 & 3 & 8 \\ -2 & 3 & -2 \end{vmatrix} &= \begin{vmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 4 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -7 \end{vmatrix} \\ &= 2(1)(-7) \\ &= -14 \end{aligned}$$

Determinants and Row Operations

Example 2

Calculate $\begin{vmatrix} 3 & 4 & 3 & -1 \\ 1 & 0 & -2 & 2 \\ -2 & 1 & 1 & 4 \\ 1 & 2 & 1 & 1 \end{vmatrix}$.

Solution

$$\begin{aligned} \begin{vmatrix} 3 & 4 & 3 & -1 \\ 1 & 0 & -2 & 2 \\ -2 & 1 & 1 & 4 \\ 1 & 2 & 1 & 1 \end{vmatrix} &= \begin{vmatrix} 0 & 4 & 9 & -7 \\ 1 & 0 & -2 & 2 \\ 0 & 1 & -3 & 8 \\ 0 & 2 & 3 & -1 \end{vmatrix} \\ &= 1(-1)^{2+1}\begin{vmatrix} 4 & 9 & -7 \\ 1 & -3 & 8 \\ 2 & 3 & -1 \end{vmatrix} \\ &= (-1)\begin{vmatrix} 0 & 21 & -39 \\ 1 & -3 & 8 \\ 0 & 9 & -17 \end{vmatrix} \\ &= (-1)(1)(-1)^{1+2}\begin{vmatrix} 21 & -39 \\ 9 & -17 \end{vmatrix} \\ &= -6 \end{aligned}$$

Determinants and Row Operations

Theorem 5.3.6

If A is an $n \times n$ matrix, then $\det A = \det A^T$.

Determinants and Row Operations

Example 1

Find the determinant of $C = \begin{bmatrix} 1 & 5 & 6 & 3 \\ 2 & 6 & 8 & -3 \\ 3 & 7 & 10 & 2 \\ 4 & 8 & 12 & 1 \end{bmatrix}$.

Solution

Applying Theorem 5.3.6 gives

$$\begin{aligned} \det C &= \det C^T \\ &= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 6 & 8 & 10 & 12 \\ 3 & -3 & 2 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \\ 3 & -3 & 2 & 1 \end{vmatrix} \\ &= 0 \end{aligned}$$

Determinants and Row Operations

Notice that Theorem 5.3.6 essentially lets us do column operations when simplifying (and only when simplifying) a determinant.

1. Adding a multiple of one column to another does not change the determinant.
2. Swapping two columns multiplies the determinant by -1.
3. Multiplying a column by a scalar multiples the determinant by the scalar.

Determinants and Row Operations

Example 2

Find the determinant of $D = \begin{bmatrix} 1 & 3 & -1 & 1 \\ -3 & 2 & 1 & 2 \\ 2 & -1 & 1 & 1 \\ 2 & -3 & 2 & -3 \end{bmatrix}$.

Solution

$$\begin{aligned}
\det D &= \begin{vmatrix} 1 & 3 & -1 & 1 \\ -3 & 2 & 1 & 2 \\ 3 & 2 & 0 & 2 \\ 2 & -3 & 2 & -3 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 3 & -1 & -2 \\ -3 & 2 & 1 & 0 \\ 3 & 2 & 0 & 0 \\ 2 & -3 & 2 & 0 \end{vmatrix} \\
&= (-2)(-1)^{1+4} \begin{vmatrix} -3 & 2 & 1 \\ 3 & 2 & 0 \\ 2 & -3 & 2 \end{vmatrix} \\
&= (-2)(-1) \begin{vmatrix} -3 & 2 & 1 \\ 3 & 2 & 0 \\ 8 & -7 & 0 \end{vmatrix} \\
&= (-2)(-1)(1)(-1)^{1+3} \begin{vmatrix} 3 & 2 \\ 8 & -7 \end{vmatrix} \\
&= (-2)(-1)(1)(1)(-37) \\
&= -74
\end{aligned}$$

Determinants and Elementary Matrices

Corollary 5.3.7

If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then $\det EA = \det E \det A$.

Proof

Observe that performing an elementary row operation multiplies the determinant of the original matrix by a non-zero real number c (in the case of swapping rows or adding a multiple of one row to another we have $c = -1$ or $c = 1$ respectively).
Since E is obtained from I by performing a single row operation, we get that $\det E = c$, where c depends on the single row operation.
Similarly, since EA is the matrix obtained from A by performing the same row operation, we have that $\det EA = c \det A = \det E \det A$. $\square$

Determinants and Elementary Matrices

Theorem 5.3.8 - Addition to the Invertible Matrix Theorem

An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.

Proof

By Theorem 5.2.5 there exists a sequence of elementary matrices $E_1, \dots, E_k$ such that $E_k \cdots E_1 A = R$ where R is the reduced row echelon form of A.
Then Corollary 5.3.7 gives

$$\det R = \det(E_k \cdots E_1 A) = \det(E_k) \cdots \det(E_1) \det A$$

Since the determinant of an elementary matrix is non-zero we get that $\det A \neq 0$ if and only if $\det R \neq 0$.
But $\det R \neq 0$ if and only if $R = I$.
Hence, $\det A \neq 0$ if and only if A is invertible. $\square$

Determinants and Elementary Matrices

Theorem 5.3.9

If A and B are $n \times n$ matrices then $\det(AB) = \det A \det B$.

Proof

By Theorem 5.2.5 there exists a sequence of elementary matrices $E_1, \dots, E_k$ such that $A = E_1^{-1} \cdots E_k^{-1} R$ where R is the reduced row echelon form of A.
If $\det A \neq 0$, then A is invertible and $R = I$.
Using Corollary 5.3.7, we get

$$\det(AB) = \det(E_1^{-1} \cdots E_k^{-1} B) = \det(E_1^{-1} \cdots E_k^{-1}) \det B = \det A \det B$$

If $\det A = 0$, then $R \neq I$ and so R contains at least one row of zeros, then Corollary 5.3.7 gives

$$\det(AB) = \det(E_1^{-1} \cdots E_k^{-1} RB) = \det E_1^{-1} \cdots \det E_k^{-1} \det(RB)$$

Observe that RB contains a row of zeros since R contains a row of zeros.
Thus, $\det(RB) = 0$ and hence

$$\det(AB) = 0 = \det A \det B$$

□

Determinants and Elementary Matrices

Corollary 5.3.10

If A is an invertible matrix, then $\det A^{-1} = \dfrac{1}{\det A}$.