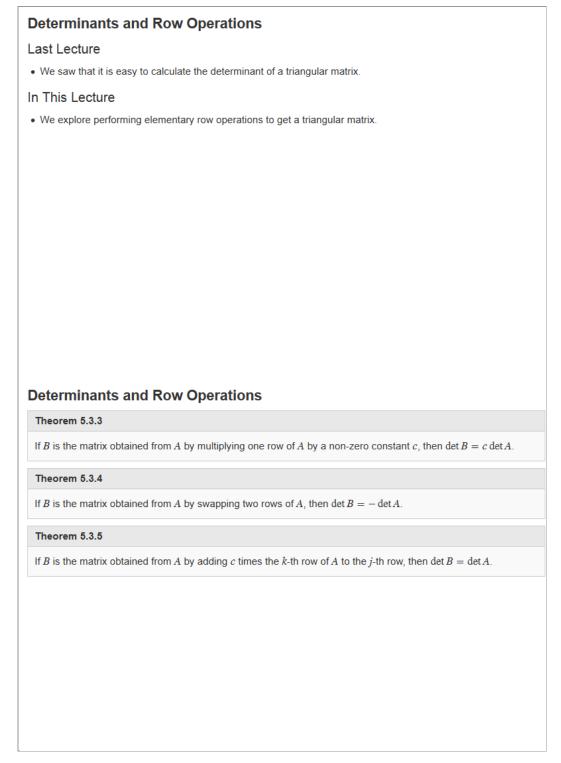
MATH 136 Module 05 Lecture 28 Course Slides

(Last Updated: January 4, 2013)



Determinants and Row Operations

Example 1

Calculate
$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 3 & 8 \\ -2 & 3 & -2 \end{vmatrix}$$

Solution

Since adding a multiple of one row to another does not change the determinant

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 3 & 8 \\ -2 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 4 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -7 \end{vmatrix}$$
$$= 2(1)(-7)$$
$$= -14$$

Determinants and Row Operations

Example 2

$$\text{Calculate} \left| \begin{array}{ccccc} 3 & 4 & 3 & -1 \\ 1 & 0 & -2 & 2 \\ -2 & 1 & 1 & 4 \\ 1 & 2 & 1 & 1 \end{array} \right|$$

Solution

$$\begin{vmatrix} 3 & 4 & 3 & -1 \\ 1 & 0 & -2 & 2 \\ -2 & 1 & 1 & 4 \\ 1 & 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 4 & 9 & -7 \\ 1 & 0 & -2 & 2 \\ 0 & 1 & -3 & 8 \\ 0 & 2 & 3 & -1 \end{vmatrix}$$
$$= 1(-1)^{2+1} \begin{vmatrix} 4 & 9 & -7 \\ 1 & -3 & 8 \\ 2 & 3 & -1 \end{vmatrix}$$
$$= (-1) \begin{vmatrix} 0 & 21 & -39 \\ 1 & -3 & 8 \\ 0 & 9 & -17 \end{vmatrix}$$
$$= (-1)(1)(-1)^{1+2} \begin{vmatrix} 21 & -39 \\ 9 & -17 \end{vmatrix}$$
$$= -6$$

Determinants and Row Operations

Theorem 5.3.6

If A is an $n \times n$ matrix, then $\det A = \det A^T$.

Determinants and Row Operations

Example 1

Find the determinant of
$$C = \begin{bmatrix} 1 & 5 & 6 & 3 \\ 2 & 6 & 8 & -3 \\ 3 & 7 & 10 & 2 \\ 4 & 8 & 12 & 1 \end{bmatrix}$$

Solution

Applying Theorem 5.3.6 gives

$$\det C = \det C^{T}$$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 6 & 8 & 10 & 12 \\ 3 & -3 & 2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \\ 3 & -3 & 2 & 1 \end{vmatrix}$$

$$= 0$$

Determinants and Row Operations

Notice that Theorem 5.3.6 essentially lets us do column operations when simplifying (and only when simplifying) a determinant.

- 1. Adding a multiple of one column to another does not change the determinant.
- 2. Swapping two columns multiplies the determinant by -1.
- 3. Multiplying a column by a scalar multiples the determinant by the scalar.

Determinants and Row Operations

Example 2

$$\text{Find the determinant of } D = \left[\begin{array}{cccc} 1 & 3 & -1 & 1 \\ -3 & 2 & 1 & 2 \\ 2 & -1 & 1 & 1 \\ 2 & -3 & 2 & -3 \end{array} \right].$$

Solution

$$\det D = \begin{vmatrix} 1 & 3 & -1 & 1 \\ -3 & 2 & 1 & 2 \\ 3 & 2 & 0 & 2 \\ 2 & -3 & 2 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & -1 & -2 \\ -3 & 2 & 1 & 0 \\ 3 & 2 & 0 & 0 \\ 2 & -3 & 2 & 0 \end{vmatrix}$$

$$= (-2)(-1)^{1+4} \begin{vmatrix} -3 & 2 & 1 \\ 3 & 2 & 0 \\ 2 & -3 & 2 \end{vmatrix}$$

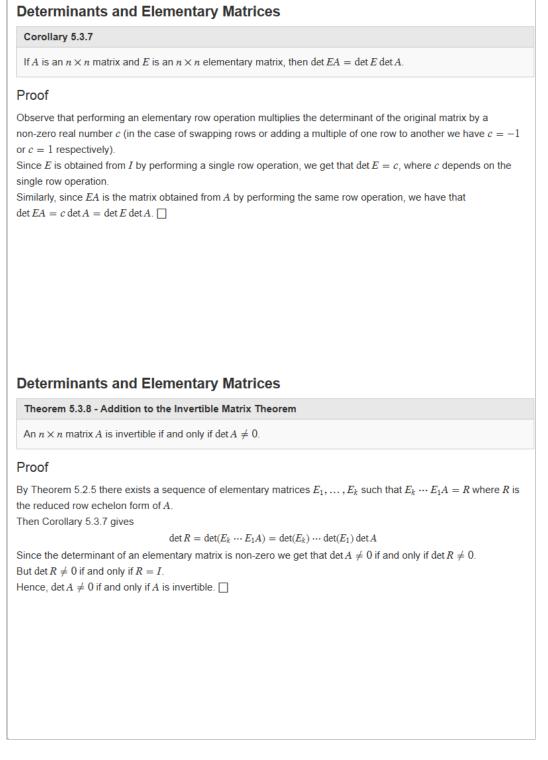
$$= (-2)(-1) \begin{vmatrix} -3 & 2 & 1 \\ 3 & 2 & 0 \\ 2 & -3 & 2 \end{vmatrix}$$

$$= (-2)(-1)(1)(-1)^{1+3} \begin{vmatrix} 3 & 2 \\ 8 & -7 & 0 \end{vmatrix}$$

$$= (-2)(-1)(1)(1)(-37)$$

$$= -74$$

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Determinants and Elementary Matrices

Theorem 5.3.9

If A and B are $n \times n$ matrices then $\det(AB) = \det A \det B$.

Proof

By Theorem 5.2.5 there exists a sequence of elementary matrices E_1, \dots, E_k such that $A = E_1^{-1} \cdots E_k^{-1} R$ where R is the reduced row echelon form of A.

If $\det A \neq 0$, then A is invertible and R = I.

Using Corollary 5.3.7, we get

$$\det(AB) = \det(E_1^{-1} \, \cdots \, E_k^{-1}B) = \det(E_1^{-1} \, \cdots \, E_k^{-1}) \det B = \det A \det B$$

If $\det A = 0$, then $R \neq I$ and so R contains at least one row of zeros, then Corollary 5.3.7 gives

$$\det(AB) = \det(E_1^{-1} \cdots E_k^{-1}RB) = \det E_1^{-1} \cdots \det E_k^{-1} \det(RB)$$

Observe that RB contains a row of zeros since R contains a row of zeros.

Thus, det(RB) = 0 and hence

$$\det(AB) = 0 = \det A \det B$$

Determinants and Elementary Matrices

Corollary 5.3.10

If A is an invertible matrix, then $\det A^{-1} = \frac{1}{\det A}$.