Diagonalization Examples

Last Lecture

- We derived an algorithm for diagonalizing a matrix or showing that a matrix is not diagonalizable.

Algorithm

To diagonalize the $n \times n$ matrix A, or show that A is not diagonalizable:

1. Find and factor the characteristic polynomial $C(\lambda) = \det(A - \lambda I)$.
2. Let $\lambda_1, \dots, \lambda_n$ denote the n-roots of $C(\lambda)$ (repeated according to multiplicity). If any of the eigenvalues λ_i are not real, then A is not diagonalizable over $\mathbb{R}$.
3. Find a basis for the eigenspace of each λ_i by finding a basis for the nullspace of $A - \lambda_i I$.
4. If $g_{\lambda_i} < a_{\lambda_i}$ for any λ_i, then A is not diagonalizable. Otherwise, form a basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$ of eigenvectors of A by combining the eigenvectors in the bases for each eigenspace of A. Let $P = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$. Then,
$$P^{-1}AP = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$
where λ_i is an eigenvalue corresponding to the eigenvector $\vec{v}_i$ for $1 \le i \le n$.

In This Lecture

- We will show several examples of this.

Diagonalization Examples

Example 1

Let $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$. Diagonalize A or show that A is not diagonalizable.

Solution

We have

$$\begin{aligned} C(\lambda) &= \det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 8 & 16 \\ 4 & 1-\lambda & 8 \\ -4 & -4 & -11-\lambda \end{vmatrix} \\ &= \begin{vmatrix} 5-\lambda & 8 & 16 \\ 0 & -3-\lambda & -3-\lambda \\ -4 & -4 & -11-\lambda \end{vmatrix} \\ &= \begin{vmatrix} 5-\lambda & 8 & 8 \\ 0 & -3-\lambda & 0 \\ -4 & -4 & -7-\lambda \end{vmatrix} \\ &= -(\lambda+3)[(\lambda+7)(\lambda-5)+32] \\ &= -(\lambda+3)(\lambda^2+2\lambda-3) \\ &= -(\lambda-1)(\lambda+3)^2 \end{aligned}$$

Hence, the eigenvalues are $\lambda_1 = 1$ with algebraic multiplicity 1 and $\lambda_2 = -3$ with algebraic multiplicity 2.

Diagonalization Examples

Example 1

Let $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$. Diagonalize A or show that A is not diagonalizable.

Solution

By theorem 6.2.3, we know that $1 \leq g_{\lambda_1} \leq a_{\lambda_1}$.
But, $a_{\lambda_1} = 1$, thus $g_{\lambda_1} = 1$.
Again by theorem 6.2.3, we know that $1 \leq g_{\lambda_2} \leq a_{\lambda_2}$. Since $a_{\lambda_2} = 2$ we could have $g_{\lambda_1} = 1$ or $g_{\lambda_2} = 2$.

Diagonalization Examples

Example 1

Let $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$. Diagonalize A or show that A is not diagonalizable.

Solution

For $\lambda_2 = -3$ we get,

$$A - \lambda_2 I = \begin{bmatrix} 8 & 8 & 16 \\ 4 & 4 & 8 \\ -4 & -4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a vector equation of the solution space for $(A - \lambda_2 I)\vec{x} = \vec{0}$ is

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}$$

So, a basis for the eigenspace E_{λ_2} of λ_2 is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Hence, the geometric multiplicity of λ_2 is also 2.

Diagonalization Examples

Example 1

Let $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$. Diagonalize A or show that A is not diagonalizable.

Solution

For $\lambda_1 = 1$ we get,

$$A - \lambda_1 I = \begin{bmatrix} 4 & 8 & 16 \\ 4 & 0 & 8 \\ -4 & -4 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a vector equation for the solution space of $(A - \lambda_1 I)\vec{x} = \vec{0}$ is

$$\vec{x} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \qquad x_3 \in \mathbb{R}$$

Thus, a basis for the eigenspace E_{λ_1} of λ_1 is $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Diagonalization Examples

Example 1

Let $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$. Diagonalize A or show that A is not diagonalizable.

Solution

We form the basis $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Consequently, if we take $P = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, then we get

$$P^{-1}AP = \text{diag}(\lambda_2, \lambda_2, \lambda_1) = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that we could have also taken: $P = \begin{bmatrix} -2 & -1 & -2 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ which would have given us

$$P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \lambda_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Diagonalization Examples

Example 2

Let $B = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix}$. Diagonalize B or show that B is not diagonalizable.

Solution

The characteristic polynomial is

$$\begin{aligned} C(\lambda) &= \begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 1-\lambda & -2 \\ -1 & 0 & -2-\lambda \end{vmatrix} \\ &= \begin{vmatrix} 3-\lambda & 1 & 1 \\ 0 & 1-\lambda & -2 \\ -3-\lambda & 0 & -2-\lambda \end{vmatrix} \\ &= (3-\lambda)(\lambda-1)(\lambda+2) - (\lambda+3)(-2-(1-\lambda)) \\ &= -(\lambda-3)(\lambda^2+\lambda-2) - (\lambda+3)(\lambda-3) \\ &= -(\lambda-3)[\lambda^2+2\lambda+1] = -(\lambda+1)^2(\lambda-3) \end{aligned}$$

Hence the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3$ with algebraic multiplicities 2 and 1 respectively.

Diagonalization Examples

Example 2

Let $B = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix}$. Diagonalize B or show that B is not diagonalizable.

Solution

For $\lambda_1 = -1$ we have

$$B - \lambda_1 I = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & -2 \\ -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

Thus, we get that

$$g_{\lambda_1} = 1 < 2 = a_{\lambda_1}$$

Consequently, B is not diagonalizable by the Diagonalization Theorem.

Diagonalization Examples

Example 3

Let $C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Diagonalize C or show that C is not diagonalizable.

Solution

The characteristic polynomial is

$$C(\lambda) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

Therefore, the eigenvalues of C are not real, so C is not diagonalizable over $\mathbb{R}$.

Diagonalization Examples

Example 4

Let $A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -1 & 0 \\ -4 & -5 & 2 \end{bmatrix}$. Diagonalize A or show that A is not diagonalizable.

Solution

We have

$$\begin{aligned} C(\lambda) &= \begin{vmatrix} 3-\lambda & 3 & -1 \\ -2 & -1-\lambda & 0 \\ -4 & -5 & 2-\lambda \end{vmatrix} \\ &= (-2)(-1)^{2+1}\begin{vmatrix} 3 & -1 \\ -5 & 2-\lambda \end{vmatrix} + (-1-\lambda)(-1)^{2+2}\begin{vmatrix} 3-\lambda & -1 \\ -4 & 2-\lambda \end{vmatrix} \\ &= (-2)(-1)(-3\lambda+1) + (-1-\lambda)(\lambda^2 - 5\lambda + 2) \\ &= -6\lambda + 2 - \lambda^3 + 4\lambda^2 + 3\lambda - 2 \\ &= -\lambda^3 + 4\lambda^2 - 3\lambda \\ &= -\lambda(\lambda-3)(\lambda-1) \end{aligned}$$

Thus, the eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 3$, and $\lambda_3 = 1$.

We see that the algebraic multiplicity of each eigenvalue is 1, hence by Corollary 6.3.4, A is diagonalizable.

Diagonalization Examples

Example 4

Let $A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -1 & 0 \\ -4 & -5 & 2 \end{bmatrix}$. Diagonalize A or show that A is not diagonalizable.

Solution

For $\lambda_1 = 0$, we have

$$A - \lambda_1 I = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -1 & 0 \\ -4 & -5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \right\}$.

For $\lambda_2 = 3$, we have

$$A - \lambda_2 I = \begin{bmatrix} 0 & 3 & -1 \\ -2 & -4 & 0 \\ -4 & -5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$.

Diagonalization Examples

Example 4

Let $A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -1 & 0 \\ -4 & -5 & 2 \end{bmatrix}$. Diagonalize A or show that A is not diagonalizable.

Solution

For $\lambda_3 = 1$, we have

$$A - \lambda_3 I = \begin{bmatrix} 0 & 3 & -1 \\ -2 & -4 & 0 \\ -4 & -5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_3} is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Thus, the matrix $P = \begin{bmatrix} -1 & -2 & -1 \\ 2 & 1 & 1 \\ 3 & 3 & 1 \end{bmatrix}$ diagonalizes A to

$$P^{-1}AP = \operatorname{diag}(0, 3, 1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Diagonalization Examples

Example 5

Let $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Diagonalize B or show that B is not diagonalizable.

Solution

We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 = -(\lambda-1)^3$$

Hence, $\lambda_1 = 1$ is an eigenvalue with $a_{\lambda_1} = 3$.
We have

$$B - \lambda_1 I = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.
Consequently, $g_{\lambda_1} = 1 < a_{\lambda_1}$, so B is not diagonalizable.