Elementary Matrices

Last Lecture

- We saw that if A is invertible, then we can solve the system of linear equations $A\vec{x} = \vec{b}$ by matrix-vector multiplication. In particular, the solution is $\vec{x} = A^{-1}\vec{b}$.

In This Lecture

- We will look at how matrix-matrix multiplication is related to elementary row operations.

Elementary Matrices

Consider a system of linear equations $A\vec{x} = \vec{b}$ where A is invertible.
By the Invertible Matrix Theorem, $A\vec{x} = \vec{b}$ is consistent.
Thus, there exists a sequence of elementary row operations such that

$$\left[A \mid \vec{b}\right] \sim \left[I \mid \vec{x}\right]$$

Notice that these are exactly the same operations that we would use to row reduce $\left[A \mid I\right]$ to $\left[I \mid A^{-1}\right]$.
Thus, these elementary row operations must be stored in A^{-1} and the matrix-vector multiplication $\vec{x} = A^{-1}\vec{b}$ must be performing the elementary row operations on $\vec{b}$ in exactly the same way as when row reducing the augmented matrix $\left[A \mid \vec{b}\right]$.
Our goal is to figure out how the elementary row operations are being stored.

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Elementary Matrices

Example 1

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

We have that

$$\left[\begin{array}{cc|c} 2 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right] \begin{array}{c} \frac{1}{2}R_1 \\ \end{array} \sim \left[\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & 4 \end{array}\right]$$

So, the solution is $\vec{x} = \begin{bmatrix} 3/2 \\ 4 \end{bmatrix}$.

We also have

$$\left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \begin{array}{c} \frac{1}{2}R_1 \\ \end{array} \sim \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$ and

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 4 \end{bmatrix}$$

Notice that the matrix $\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$ has been obtained from the identity matrix by performing the single row operation $\frac{1}{2}R_1$.

Also notice that multiplying this matrix by $\vec{b}$ had the same effect as performing the same row operation on $\vec{b}$.

Elementary Matrices

Example 2

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let E be the matrix obtained from the 2×2 identity matrix by performing the single row operation kR_1 with $k \neq 0$. Determine EA.

Solution

Multiplying the first row of the 2×2 identity matrix by k gives

$$E = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, we get

$$EA = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$$

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Elementary Matrices

Definition: An elementary matrix is an $n \times n$ matrix that is created by performing a single elementary row-operation on the $n \times n$ identity matrix.

Theorem 5.2.1

If A is an $m \times n$ matrix and E is the $m \times m$ elementary matrix corresponding to the row operation $R_i + cR_j$, for $i \neq j$, then EA is the matrix obtained from A by performing the elementary row operation $R_i + cR_j$ on A.

Theorem 5.2.2

If A is an $m \times n$ matrix and E is the $m \times m$ elementary matrix corresponding to the row operation cR_i, $c \neq 0$, then EA is the matrix obtained from A by performing the elementary row operation cR_i on A.

Theorem 5.2.3

If A is an $m \times n$ matrix and E is the $m \times m$ elementary matrix corresponding to the row operation $R_i \leftrightarrow R_j$, for $i \neq j$, then EA is the matrix obtained from A by performing the elementary row operation $R_i \leftrightarrow R_j$ on A.

Elementary Matrices

Since elementary row operations do not change the rank of a matrix, we immediately get the following result.

Corollary 5.2.4

If A is an $m \times n$ matrix and E is an $m \times m$ elementary matrix, then

$$\operatorname{rank}(EA) = \operatorname{rank} A$$

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Elementary Matrices

Example 1

Find the 2×2 elementary matrix corresponding to each row operation by performing the elementary row operation on the 2×2 identity matrix.

(a) $4R_2$

Solution

We have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} 4R_2 \sim \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

Thus, the elementary matrix corresponding to $4R_2$ is $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$.

(b) $R_2 - 3R_1$

Solution

We have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_2 - 3R_1 \sim \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

Thus, the elementary matrix corresponding to $R_2 - 3R_1$ is $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$.

Elementary Matrices

Example 1

Find the 2×2 elementary matrix corresponding to each row operation by performing the elementary row operation on the 2×2 identity matrix.

(c) $R_1 \leftrightarrow R_2$

Solution

We have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_1 \leftrightarrow R_2 \sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus, the elementary matrix corresponding to $R_1 \leftrightarrow R_2$ is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

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Elementary Matrices

Example 2

Determine which of the following matrices are elementary. For each elementary matrix, indicate the associated elementary row operation.

(a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Solution

Notice that it would take the two elementary row operations $R_1 + R_2$ and then $R_2 + R_3$ to get this matrix from I, so it is not elementary.

(b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution

$0R_1$ is not an elementary row operation.
Hence, the matrix is not an elementary matrix.

Elementary Matrices

Example 2

Determine which of the following matrices are elementary. For each elementary matrix, indicate the associated elementary row operation.

(c) $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

Solution

Observe that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \overset{R_1 + 3R_2}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

So, this is the 2×2 elementary matrix corresponding to $R_1 + 3R_2$.

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Elementary Matrices

Since elementary row operations are all reversible, all elementary matrices are invertible.
In particular, the inverse of an elementary matrix is the elementary matrix associated with the inverse elementary row operations.

Elementary Matrices

Example 3

Find the inverse of each of the following elementary matrices. Check your answer by multiplying the matrices together.

(a) $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution

The elementary row operation corresponding to E_1 is $R_2 - 4R_1$.
To row reduce E_1 back to the identity matrix we would apply the elementary row operation $R_2 + 4R_1$.
Thus, the inverse of E_1 is the elementary matrix associated with $R_2 + 4R_1$.
That is,

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Indeed, we have

$$E_1 E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Elementary Matrices

Example 3

Find the inverse of each of the following elementary matrices. Check your answer by multiplying the matrices together.

(b) $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution

E_2 corresponds to $-\frac{1}{2}R_2$.
To row reduce this matrix back to the identity matrix we would apply $-2R_2$.
Thus,

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We have

$$E_2^{-1}E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as required.

Matrix Decomposition

Theorem 5.2.5

If A is an $m \times n$ matrix with reduced row echelon form R, then there exists a sequence $E_1, \dots, E_k$ of $m \times m$ elementary matrices such that $E_k \cdots E_2E_1A = R$. In particular,

$$A = E_1^{-1}E_2^{-1} \cdots E_k^{-1}R$$

Proof

Using the methods of Module 2, we know that we can row reduce a matrix A to its RREF R with a sequence of elementary row operations.
Let E_1 denote the elementary matrix corresponding to the first row operation performed, E_2 the elementary matrix corresponding to the second row operation performed, up to E_k corresponding to the last row operation performed.
Then, Theorems 5.2.1, 5.2.2, and 5.2.3, give us that

$$E_kE_{k-1} \cdots E_2E_1A = R \tag{1}$$

Because each elementary matrix is invertible, we get that the product $E_kE_{k-1} \cdots E_1$ is invertible by Theorem 5.1.6 (2).
In particular, multiplying by the inverse of $E_k \cdots E_1$ on both sides of (1) we get that

$$A = (E_k \cdots E_1)^{-1}R = E_1^{-1}E_2^{-1} \cdots E_k^{-1}R$$

as required. □

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Matrix Decomposition

Example

Let $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -2 & -3 & 1 & 0 \\ 1 & 0 & -2 & 5 \end{bmatrix}$. Write A as a product of elementary matrices and its RREF.

Solution

We row reduce A to its RREF keeping track of the elementary row operations used.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ -2 & -3 & 1 & 0 \\ 1 & 0 & -2 & 5 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_1 \\ R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -2 & 5 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ R_3 + 2R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{matrix} \\ \\ \frac{1}{5}R_3 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, our elementary matrices are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \ E_3 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \ E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

Matrix Decomposition

Example

Let $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -2 & -3 & 1 & 0 \\ 1 & 0 & -2 & 5 \end{bmatrix}$. Write A as a product of elementary matrices and its RREF.

Solution

Then, Theorem 5.2.5 says that

$$\begin{aligned} A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}R \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

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Matrix Decomposition

If A is invertible, then the reduced row echelon form of A is I.
Thus, Theorem 5.2.5 tells us that there exists a sequence of elementary matrices $E_1, \dots, E_k$ such that

$$E_k \cdots E_2 E_1 A = I$$

Then, the product $E_k \cdots E_1$ is a left inverse of A and hence

$$A^{-1} = E_k \cdots E_1$$

Theorem 5.2.5 also tells us that we have

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Corollary 5.2.6

If A is an invertible matrix, then A and A^{-1} can be written as a product of elementary matrices.

Matrix Decomposition

Example

Let $A = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix}$. Write A and A^{-1} as a product of elementary matrices.

Solution

We first row reduce A to I keeping track of the elementary row operations used.

$$\begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix} \begin{matrix} R_2 + 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{matrix} \frac{1}{3} R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, we have

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$$

Hence, $E_3 E_2 E_1 A = I$, and so $A^{-1} = E_3 E_2 E_1$.
We find that

$$E_3 E_2 E_1 = E_3 \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1/3 & 2/3 \end{bmatrix}$$

Also, we have

$$A = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

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