Homogeneous Systems and Rank

In This Lecture

- We will focus more on the theory of solving systems of linear equations.

Homogeneous Systems

Definition: A system of linear equations is said to be a homogeneous system if the right-hand side only contains zeros. That is, it has the form $\left[A \mid \vec{0}\right]$.

Homogeneous Systems

Example

Determine if $\left\{\begin{bmatrix}1\\1\\0\\2\end{bmatrix},\begin{bmatrix}2\\3\\2\\1\end{bmatrix},\begin{bmatrix}1\\2\\2\\-1\end{bmatrix},\begin{bmatrix}-1\\1\\7\\8\end{bmatrix}\right\}$ is linearly independent or dependent.

Solution

Consider

$$c_1\begin{bmatrix}1\\1\\0\\2\end{bmatrix}+c_2\begin{bmatrix}2\\3\\2\\1\end{bmatrix}+c_3\begin{bmatrix}1\\2\\2\\-1\end{bmatrix}+c_4\begin{bmatrix}-1\\1\\7\\8\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1+2c_2+c_3-c_4\\c_1+3c_2+2c_3+c_4\\2c_2+2c_3+7c_4\\2c_1+c_2-c_3+8c_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Comparing entries gives the homogeneous system

$$\begin{aligned}c_1+2c_2+c_3-c_4&=0\\c_1+3c_2+2c_3+c_4&=0\\2c_2+2c_3+7c_4&=0\\2c_1+c_2-c_3+8c_4&=0\end{aligned}$$

Homogeneous Systems

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$$\left[\begin{array}{cccc|c}1&2&1&-1&0\\1&3&2&1&0\\0&2&2&7&0\\2&1&-1&8&0\end{array}\right]\sim\left[\begin{array}{cccc|c}1&0&-1&0&0\\0&1&1&0&0\\0&0&0&1&0\\0&0&0&0&0\end{array}\right]$$

Since there is a free variable, there are infinitely many solutions and hence the set is linearly dependent.

Homogeneous Systems

We notice some important things:

1. In some applications we will not actually need to find solutions. We may just be interested in determining if a system is inconsistent, consistent with a unique solution, or consistent with infinitely many solutions.
2. $\vec{x} = \vec{0}$ is always a solution to a homogeneous system. Thus, every homogeneous system is consistent.
3. When solving a homogeneous system, we can just row reduce the coefficient matrix since the right hand side will always only contain 0s.

Homogeneous Systems

Theorem 2.2.3

The solution set of a homogeneous system of m linear equations in n variables is a subspace of $\mathbb{R}^n$.

Definition: The solution set of a homogeneous system is called the solution space of the system.

Homogeneous Systems

Example

Find the solution space of the homogeneous system

$$\begin{aligned} 2x_1 + 5x_2 - 3x_3 &= 0 \\ x_1 - 3x_2 + 4x_3 &= 0 \end{aligned}$$

Solution

As mentioned above, since the system is homogeneous we just row reduce the coefficient matrix to RREF. We get

$$\begin{bmatrix} 2 & 5 & -3 \\ 1 & -3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Rewriting the RREF back as a homogeneous system, we get

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

Since x_3 is a free-variable, we get $x_1 = -x_3$ and $x_2 = x_3$.
Therefore, every vector $\vec{x}$ in the solution space has the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad x_3 \in \mathbb{R}$$

Observe that the solution space indeed is a line through the origin in $\mathbb{R}^3$.

Rank

Definition: The rank of a matrix A is the number of leading ones in the RREF of the matrix and is denoted $\operatorname{rank} A$.

Example

Determine the rank of the matrix $A = \begin{bmatrix} 1 & 3 & 3 & 3 & 0 \\ 2 & 5 & 4 & 4 & 0 \\ 0 & 2 & 4 & 8 & 4 \\ 1 & 4 & 5 & -3 & -8 \end{bmatrix}$.

Solution

Row reducing, we find that the RREF of A is

$$R = \begin{bmatrix} 1 & 0 & -3 & 0 & 3 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since R has 3 leading 1s, the rank of A is 3.

Rank

Since the rank of a matrix is the number of leading ones in the RREF of the matrix, and it is clear that we cannot have more leading ones than the number of rows or columns in the matrix, we get:

Theorem 2.2.4

If A is a matrix with m rows and n columns, then $\operatorname{rank} A \leq \min(m, n)$.

Theorem 2.2.5 (a proof is in the course notes)

Let A be the coefficient matrix of a system of m linear equations in n unknowns $\left[A \mid \vec{b}\right]$.

1. If the rank of A is less than the rank of the augmented matrix $\left[A \mid \vec{b}\right]$, then the system is inconsistent.
2. If the system $\left[A \mid \vec{b}\right]$ is consistent, then the system contains $(n - \operatorname{rank} A)$ free variables (parameters).
3. $\operatorname{rank} A = m$ if and only if the system $\left[A \mid \vec{b}\right]$ is consistent for every $\vec{b} \in \mathbb{R}^m$.

Part (2) says that a consistent system of m linear equations in n variables with coefficient matrix A has a unique solution if and only if $\operatorname{rank} A = n$.

Rank

The next theorem demonstrates that if $\left[A \mid \vec{b}\right]$ is consistent with $\operatorname{rank} A < n$, then the set of $(n - \operatorname{rank} A)$ vectors corresponding to the free variables in our method for finding the general solution is necessarily linearly independent.

Theorem 2.2.6

Let $\left[A \mid \vec{b}\right]$ be a consistent system of m linear equations in n variables with RREF $\left[R \mid \vec{c}\right]$. If $\operatorname{rank} A = k \leq n$, then a vector equation of the solution set of $\left[A \mid \vec{b}\right]$ has the form

$$\vec{x} = \vec{d} + t_1\vec{v}_1 + \cdots + t_{n-k}\vec{v}_{n-k}, \qquad t_1, \ldots, t_{n-k} \in \mathbb{R}$$

where $\vec{d} \in \mathbb{R}^n$ and $\{\vec{v}_1, \ldots, \vec{v}_{n-k}\}$ is a linearly independent set in $\mathbb{R}^n$. In particular, the solution set of $\left[A \mid \vec{b}\right]$ is an $(n-k)$-flat in $\mathbb{R}^n$.