Matrix Mappings

In This Lecture

- We will use matrix-vector multiplication to define an important class of functions.

Matrix Mappings

Recall that a function is a rule associates each element $a \in A$ with a unique element $f(a) \in B$.
We write $f : A \to B$ and we call A the domain, and B the codomain.
The value of $f(a)$ is called the image of a under f.

Definition: For any $A \in M_{m\times n}(\mathbb{R})$ a function $f : \mathbb{R}^n \to \mathbb{R}^m$ defined by $f(\vec{x}) = A\vec{x}$ is called a matrix mapping.

Note

If $f : \mathbb{R}^n \to \mathbb{R}^m$, then we should write

$$f\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

However, we will often write

$$f(x_1, \dots, x_n) = (y_1, \dots, y_m)$$

instead.

Matrix Mappings

Example 1

Let $A = \begin{bmatrix} 2 & 3 \\ -4 & 0 \\ 5 & 1 \end{bmatrix}$ and define $f(\vec{x}) = A\vec{x}$.

(a) What is the domain and codomain of f?

Solution

For $A\vec{x}$ to be defined, $\vec{x}$ must have 2 rows since A has 2 columns.
Hence, the domain of f is $\mathbb{R}^2$.
Also, since A has 3 rows, the product $A\vec{x}$ will have 3 rows.
Hence, the codomain of f is $\mathbb{R}^3$.
We write $f : \mathbb{R}^2 \to \mathbb{R}^3$.

(b) Calculate $f(\vec{v})$ where $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Solution

We have

$$f(\vec{v}) = A\vec{v} = \begin{bmatrix} 2 & 3 \\ -4 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \\ 9 \end{bmatrix}$$

We may write $f(2, -1) = (1, -8, 9)$.

Matrix Mappings

Example 1

Let $A = \begin{bmatrix} 2 & 3 \\ -4 & 0 \\ 5 & 1 \end{bmatrix}$ and define $f(\vec{x}) = A\vec{x}$.

(c) Determine $f(\vec{x})$ for any $\vec{x} \in \mathbb{R}^2$.

Solution

We have

$$f(\vec{x}) = A\vec{x} = \begin{bmatrix} 2 & 3 \\ -4 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ -4x_1 + 0x_2 \\ 5x_1 + x_2 \end{bmatrix}$$

We may write $f(x_1, x_2) = (2x_1 + 3x_2, -4x_1, 5x_1 + x_2)$.

Matrix Mappings

Example 2

Let $B = \begin{bmatrix} 3 & -2 \\ 4 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}$ and let $g(\vec{x}) = B\vec{x}$ be the corresponding matrix mapping.

(a) What is the domain and codomain of g?

Solution

For $B\vec{x}$ to be defined, $\vec{x}$ must have 2 rows since B has 2 columns.
Thus, the domain of g is $\mathbb{R}^2$.
Since B has 4 rows, the product $B\vec{x}$ will also have 4 rows.
Hence, the codomain of g is $\mathbb{R}^4$.

Matrix Mappings

Example 2

Let $B = \begin{bmatrix} 3 & -2 \\ 4 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}$ and let $g(\vec{x}) = B\vec{x}$ be the corresponding matrix mapping.

(b) Determine $g(1,0)$, $g(0,1)$, and $g(x_1, x_2)$.

Solution

We have

$$g(1,0) = B\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \\ -1 \end{bmatrix}$$

$$g(0,1) = B\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$g(x_1,x_2) = B\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_2 \\ 4x_1 + x_2 \\ 0x_1 + 0x_2 \\ -x_1 + 0x_2 \end{bmatrix}$$

Matrix Mappings

Example 2

Let $B = \begin{bmatrix} 3 & -2 \\ 4 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}$ and let $g(\vec{x}) = B\vec{x}$ be the corresponding matrix mapping.

Important Observations

We first observe that $B = \begin{bmatrix} g(1,0) & g(0,1) \end{bmatrix}$.

$$g(x_1, x_2) = \begin{bmatrix} g(1,0) & g(0,1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 g(1,0) + x_2 g(0,1) = x_1 \begin{bmatrix} 3 \\ 4 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Finally we observe that if A is an $m \times n$ matrix and $g(\vec{x}) = A\vec{x}$, then the domain of g is $\mathbb{R}^n$ and the codomain of g is $\mathbb{R}^m$.

Linearity of Matrix Mappings

Theorem 3.2.1

Let A be an $m \times n$ matrix and let $f(\vec{x}) = A\vec{x}$. Then, for any vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$ we have

$$f(s\vec{x} + t\vec{y}) = sf(\vec{x}) + tf(\vec{y})$$

Proof

Using properties of operations on matrices we get

$$f(s\vec{x} + t\vec{y}) = A(s\vec{x} + t\vec{y}) = sA\vec{x} + tA\vec{y} = sf(\vec{x}) + tf(\vec{y})$$

□

Notice that a function which has this property will in fact **preserve linear combinations**, that is

$$f(t_1\vec{x}_1 + \cdots + t_k\vec{x}_k) = t_1 f(\vec{x}_1) + \cdots + t_k f(\vec{x}_k)$$

We will call this property the linearity property.