Powers of Matrices

In This Lecture

- We will look at a simple application of diagonalization.

Powers of Matrices

Let A be an $n \times n$ matrix.
We define

$$\begin{aligned} A^2 &= A(A) \\ A^3 &= A \times A \times A = A^2(A) \\ A^{k+1} &= A^k A \qquad \text{for any positive integer } k \end{aligned}$$

Consider $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

How can we calculate large powers of A by hand?

If $D = \text{diag}(2, 3) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, then

$$\begin{aligned} D^2 &= DD = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \\ D^3 &= D^2 D = \begin{bmatrix} 8 & 0 \\ 0 & 27 \end{bmatrix} \\ D^4 &= D^3 D = \begin{bmatrix} 16 & 0 \\ 0 & 81 \end{bmatrix} \end{aligned}$$

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Lemma 6.4.1

If $D = \text{diag}(d_1, \dots, d_n)$, then $D^m = \text{diag}(d_1^m, \dots, d_n^m)$ for any positive integer m.

Proof

If $m = 1$, the result is clearly true. Assume the result is true for $m = k$.
Then,

$$D^{k+1} = D^k D = \text{diag}(d_1^k, \dots, d_n^k)\,\text{diag}(d_1, \dots, d_n) = \text{diag}(d_1^{k+1}, \dots, d_n^{k+1})$$

as required. □

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Theorem 6.4.2

If there exists an invertible matrix P such that $P^{-1}AP = D$ is diagonal, then

$$A^m = PD^mP^{-1}$$

for any positive integer m.

Proof

We first observe that if $P^{-1}AP = D$, then we can multiply on the left by P and on the right by P^{-1} to get $A = PDP^{-1}$.
If $m = 1$, then we have $A^1 = PD^1P^{-1}$ as required.
Assume the result is true for some $m = k$. That is, assume that $A^k = PD^kP^{-1}$.
Then,

$$\begin{aligned} A^{k+1} &= A^kA \\ &= (PD^kP^{-1})(PDP^{-1}) \\ &= PD^k(P^{-1}P)DP^{-1} \\ &= PD^kIDP^{-1} \\ &= PD^{k+1}P^{-1} \end{aligned}$$

by Lemma 6.4.1. □

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Example 1

Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. Find A^{1000}.

Solution

We first diagonalize A. We have

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$$

Thus, the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 3$.

For $\lambda_1 = 2$ we get

$$A - \lambda_1 I = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 3$ we get

$$A - \lambda_2 I = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

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Example 1

Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. Find A^{1000}.

Solution

Therefore, taking $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ gives

$$P^{-1}AP = D$$

where $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Thus, by Theorem 6.4.2 we get that

$$\begin{aligned} A^{1000} &= PD^{1000}P^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{1000} & 0 \\ 0 & 3^{1000} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2^{1001} & 3^{1000} \\ 2^{1000} & 3^{1000} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2^{1001} - 3^{1000} & -2^{1001} + 2 \cdot 3^{1000} \\ 2^{1000} - 3^{1000} & -2^{1000} + 2 \cdot 3^{1000} \end{bmatrix} \end{aligned}$$

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Example 2

Let $B = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. Calculate B^{100}.

Solution

We first diagonalize B. We have

$$\begin{aligned} C(\lambda) = \det(B - \lambda I) &= \begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} \\ &= \begin{vmatrix} 2-\lambda & 0 & 2 \\ 2 & -\lambda & 2 \\ 2 & \lambda & 2-\lambda \end{vmatrix} \\ &= \begin{vmatrix} 2-\lambda & 0 & 2 \\ 2 & -\lambda & 2 \\ 4 & 0 & 4-\lambda \end{vmatrix} \\ &= -\lambda(\lambda^2 - 6\lambda) \\ &= -\lambda^2(\lambda - 6) \end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 6$.

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Example 2

Let $B = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. Calculate B^{100}.

Solution

For $\lambda_1 = 0$ we get

$$B - \lambda_1 I = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 6$ we get

$$B - \lambda_2 I = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

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Example 2

Let $B = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. Calculate B^{100}.

Solution

Therefore, we have $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

So, by Theorem 6.4.2 we get that

$$\begin{aligned} B^{100} = PD^{100}P^{-1} &= \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 6^{100}/3 & 6^{100}/3 & 6^{100}/3 \end{bmatrix} \\ &= 2 \cdot 6^{99} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{aligned}$$