Solving Systems of Linear Equations

Last Lecture

- We learned how to represent systems of linear equations in matrix notation.

In This Lecture

- We will convert the operations used in substitution and elimination into operations on the rows of a matrix and use these to solve systems.
- We will call these operations elementary row operations (EROs).

Elementary Row Operations (EROs)

1. Multiply a row by a non-zero number.
2. Add a multiple of one row to another row.
3. Swap one row with another row.

Theorem 2.2.1

If the augmented matrix $\left[A_2 \mid \vec{b}_2\right]$ can be obtained from the augmented matrix $\left[A_1 \mid \vec{b}_1\right]$ by performing elementary row operations, then the corresponding systems of linear equations are equivalent.

Definition: Two matrices A and B are said to be row equivalent if there exists a sequence of elementary row operations that transform A into B. We write $A \sim B$. The procedure of applying elementary row operations to transform A into B is called row reducing.

Notice that every elementary row operation is reversible. Hence, if A is row equivalent to B, then B is row equivalent to A.

DO NOT use an operation of the form add a mutliple of one row to a mutliple of another. This is **not** an elementary row operation.

Elementary Row Operations (EROs)

We also invent some notation for our elementary row operations. We write:

- cR_i to indicate multiplying the i-th row by $c \neq 0$.
- $R_i + cR_j$ to indicate adding c times the j-th row to the i-th row.
- $R_i \leftrightarrow R_j$ to indicate swapping the i-th row and the j-th row.

Elementary Row Operations (EROs)

Example

Write the augmented matrix for the following system and row reduce to get a system that is easier to solve. Find the solution set.

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 0x_1 + 2x_2 - 6x_3 &= 2 \\ 3x_1 + 6x_2 - 5x_3 &= 4 \end{aligned}$$

Solution

The augmented matrix for the system is $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & -6 & 2 \\ 3 & 6 & -5 & 4 \end{array}\right]$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & -6 & 2 \\ 3 & 6 & -5 & 4 \end{array}\right] \begin{array}{c} \\ \\ R_3 - 3R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & -6 & 2 \\ 0 & 3 & -8 & 1 \end{array}\right] \begin{array}{c} \\ \frac{1}{2}R_2 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 3 & -8 & 1 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \\ R_3 - 3R_2 \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} R_1 - 4R_3 \\ R_2 + 3R_3 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The new system of linear equations is $x_1 = 8, x_2 = -5, x_3 = -2$.

This procedure is called Gauss-Jordan elimination.

It can be worth trying to avoid fractions for as long as possible.

Mostly, the only way to become fast and efficient at row-reducing is to row-reduce lots and lots of matrices!

Reduced Row Echelon Form

A matrix R is said to be in reduced row echelon form (RREF) if:

1. All rows containing a non-zero entry are above rows which only contains zeros.
2. The first non-zero entry in each non-zero row is 1, called a leading one (or a pivot).
3. The leading one in each non-zero row is to the right of the leading one in any row above it.
4. A leading one is the only non-zero entry in its column.

If a matrix A is row equivalent to R, we say that R is the reduced row echelon form of A.

Theorem 2.2.2

Every matrix has a unique reduced row echelon form.

Examples of Solving Systems

Example 1

Find the solution set of the system

$$\begin{aligned} 2x_1 - x_2 + 2x_3 &= 1 \\ x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 + 3x_2 + 3x_3 &= 5 \end{aligned}$$

Solution

We row reduce the corresponding augmented matrix to RREF.

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 3 & 3 & 3 & 5 \end{array}\right] \begin{array}{c} \\ R_1 \leftrightarrow R_2 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & -1 & 2 & 1 \\ 3 & 3 & 3 & 5 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -5 & 0 & -5 \\ 0 & -3 & 0 & -4 \end{array}\right] \begin{array}{c} \\ -\frac{1}{5}R_2 \\ \\ \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & -3 & 0 & -4 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ \\ R_3 + 3R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right] \begin{array}{c} \\ \\ (-1)R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 - R_3 \\ R_2 - R_3 \\ \\ \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

If we rewrite this as a system of linear equations we get $x_1 + x_3 = 0$, $x_2 = 0$, and $0 = 1$.

Hence, the system is inconsistent and so the solution set is empty.

Examples of Solving Systems

Example 2

Find all solutions of the system

$$\begin{aligned} x_1 + 3x_2 &= 1 \\ x_1 + x_2 &= 0 \\ 2x_1 + 8x_2 &= 3 \end{aligned}$$

Solution

Row reducing the corresponding augmented matrix to RREF gives

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 1 & 1 & 0 \\ 2 & 8 & 3 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array} \sim \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{array}\right] -\tfrac{1}{2}R_2 \sim \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & 1/2 \\ 0 & 2 & 1 \end{array}\right] \begin{array}{c} R_1 - 3R_2 \\ \\ R_3 - 2R_2 \end{array} \sim \left[\begin{array}{cc|c} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right]$$

Convert the RREF into equation form gives $x_1 = -1/2$ and $x_2 = 1/2$.

Thus, the system is consistent with unique solution $\vec{x} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$.

Examples of Solving Systems

Example 3

Find all solutions to the system

$$\begin{aligned} x_2 + 2x_3 &= 1 \\ x_1 + 2x_2 + 3x_3 &= 2 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

Solution

Row reducing the corresponding augmented matrix to RREF gives

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right] \begin{array}{c} \\ \\ R_3 \leftrightarrow R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right] R_2 - R_1 \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \\ R_3 - R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Converting this back to equation form gives

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 2x_3 &= 1 \end{aligned}$$

We can rewrite these equations as

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= 1 - 2x_3 \end{aligned}$$

Examples of Solving Systems

Example 3

Find all solutions to the system

$$\begin{aligned} x_2 + 2x_3 &= 1 \\ x_1 + 2x_2 + 3x_3 &= 2 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

Solution

We can rewrite these equations as

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= 1 - 2x_3 \end{aligned}$$

Thus, all solution $\vec{x}$ of the system have the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 1 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

where $x_3 \in \mathbb{R}$.

Free Variables

Definition: Any variable whose column does not contain a leading one in the RREF of the coefficient matrix of a system of linear equations is called a free variable.

If we find that a system of linear equations has one or more free variables and is consistent, then the system has infinitely many solutions and each free variable will correspond to a parameter in the general solution of the system.

In particular, after converting the RREF of the augmented matrix back to equation form, we always move all free-variables to the right side of the equation and replace them with parameters.

Free Variables

Example

Solve the system

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 1 \\ x_3 - 2x_4 &= 2 \end{aligned}$$

Solution

Row reducing the augmented matrix gives

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 & 2 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & -1 \\ 0 & 0 & 1 & -2 & 2 \end{array}\right]$$

$$\begin{aligned} x_1 + 2x_2 + 3x_4 = -1 &\Rightarrow x_1 = -1 - 2x_2 - 3x_4 \\ x_3 - 2x_4 = 2 &\Rightarrow x_3 = 2 + 2x_4 \end{aligned}$$

Hence, every solution $\vec{x}$ has the form

$$\begin{aligned} \vec{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 - 2x_2 - 3x_4 \\ x_2 \\ 2 + 2x_4 \\ x_4 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix} \end{aligned}$$

for $x_2, x_4 \in \mathbb{R}$.