## Spanning and Linear Independence

### Previously

- We have looked at vector spaces and subspaces of vector spaces.

### In This Lecture

- We extend the definitions of spanning and linear independence to general vector spaces.

## Spanning

**Definition:** Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in a vector space $\mathbb{V}$. The span of $\mathcal{B}$ is defined by

$$\operatorname{Span} \mathcal{B} = \{c_1\vec{v}_1 + \cdots + c_k\vec{v}_k \mid c_1, \dots, c_k \in \mathbb{R}\}$$

## Spanning

### Example 1

Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix} \right\}$. Determine if $A = \begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix}$ is in Span $\mathcal{B}$.

**Solution**

We need to determine if there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$\begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} = c_1 \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} + c_3 \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2c_1 + c_2 + 3c_3 & 3c_1 + 2c_2 + 4c_3 \\ c_1 - c_2 + c_3 & 2c_1 + 2c_2 - c_3 \end{bmatrix}$$

Comparing entries we get the system of linear equations

$$\begin{aligned} 2c_1 + c_2 + 3c_3 &= 2 \\ 3c_1 + 2c_2 + 4c_3 &= 2 \\ c_1 - c_2 + c_3 &= 6 \\ 2c_1 + 2c_2 - c_3 &= 3 \end{aligned}$$

## Spanning

### Example 1

Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix} \right\}$. Determine if $A = \begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix}$ is in Span $\mathcal{B}$.

**Solution**

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 2 \\ 3 & 2 & 4 & 2 \\ 1 & -1 & 1 & 6 \\ 2 & 2 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, the system is consistent and so $A \in \operatorname{Span} \mathcal{B}$.

In particular, we have

$$\begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} = 4 \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}$$

## Spanning

### Example 2

Prove that the subspace $\mathbb{S} = \{a + bx + cx^2 \in P_2(\mathbb{R}) \mid a + b = c\}$ of $P_2(\mathbb{R})$ is spanned by $\mathcal{B} = \{1 + x^2, 1 - x\}$.

**Solution**

Let $p(x) \in \mathbb{S}$. Then, we have that $p(x) = a + bx + cx^2$ where $a + b = c$.
Hence, we can write $p(x) = a + bx + (a + b)x^2$.
To show $\operatorname{Span} \mathcal{B} = \mathbb{S}$ we technically need to show that $\operatorname{Span} \mathcal{B} \subseteq \mathbb{S}$ and $\mathbb{S} \subseteq \operatorname{Span} \mathcal{B}$.
However, it is easy that every vector in $\mathcal{B}$ belongs to $\mathbb{S}$ and therefore since $\mathbb{S}$ is closed under linear combinations $\operatorname{Span} \mathcal{B} \subseteq \mathbb{S}$.

## Spanning

### Example 2

Prove that the subspace $\mathbb{S} = \{a + bx + cx^2 \in P_2(\mathbb{R}) \mid a + b = c\}$ of $P_2(\mathbb{R})$ is spanned by $\mathcal{B} = \{1 + x^2, 1 - x\}$.

**Solution**

Let $p(x) \in \mathbb{S}$. Then, we have that $p(x) = a + bx + cx^2$ where $a + b = c$.
Hence, we can write $p(x) = a + bx + (a + b)x^2$.
Consider

$$a + bx + (a + b)x^2 = c_1(1 + x^2) + c_2(1 - x) = (c_1 + c_2) - c_2 x + c_1 x^2$$

Comparing coefficients of like powers of $x$ we get the system

$$\begin{aligned} c_1 + c_2 &= a \\ -c_2 &= b \\ c_1 &= a + b \end{aligned}$$

We see that the system has solution $c_1 = a + b$ and $c_2 = -b$.
Therefore, we have that

$$a + bx + (a + b)x^2 = (a + b)(1 + x^2) - b(1 - x) \qquad (1)$$

Consequently, $\mathbb{S} = \operatorname{Span} \mathcal{B}$.

## Spanning

**Theorem 4.1.3**

If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a set of vectors in a vector space $\mathbb{V}$, then $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ is a subspace of $\mathbb{V}$.

**Theorem 4.1.4**

Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in a vector space $\mathbb{V}$. If $\vec{v}_i \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$, then

$$\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\} = \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$$

## Linear Independence

**Definition:** Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in a vector space $\mathbb{V}$. If

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$$

has a solution such that some $c_i \neq 0$, then $\mathcal{B}$ is said to be linearly dependent. If the only solution is $c_1 = \cdots = c_k = 0$, then $\mathcal{B}$ is said to be linearly independent.

## Linear Independence

### Example 1

Determine if the set $\{1 + x - 2x^2, 1 - 2x + x^2, -2 + x + x^2\}$ in $P_2(\mathbb{R})$ is linearly independent or linearly dependent.

**Solution**

Consider

$$\begin{aligned} 0 &= c_1(1 + x - 2x^2) + c_2(1 - 2x + x^2) + c_3(-2 + x + x^2) \\ 0 + 0x + 0x^2 &= (c_1 + c_2 - 2c_3) + (c_1 - 2c_2 + c_3)x + (-2c_1 + c_2 + c_3)x^2 \end{aligned}$$

Comparing like powers of $x$ we get the homogeneous system

$$\begin{aligned} c_1 + c_2 - 2c_3 &= 0 \\ c_1 - 2c_2 + c_3 &= 0 \\ -2c_1 + c_2 + c_3 &= 0 \end{aligned}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the system has infinitely many solutions and so the system is linearly dependent.

## Linear Independence

### Example 2

Determine if the set $\left\{ \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -5 \\ 6 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 1 & -4 \end{bmatrix} \right\}$ in $M_{2\times 2}(\mathbb{R})$ is linearly independent or linearly dependent.

**Solution**

Consider

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 & -5 \\ 6 & 4 \end{bmatrix} + c_3 \begin{bmatrix} 3 & 1 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 + 3c_3 & -2c_1 - 5c_2 + c_3 \\ 3c_1 + 6c_2 + c_3 & c_1 + 4c_2 - 4c_3 \end{bmatrix}$$

Comparing entries we get the homogeneous system

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ -2c_1 - 5c_2 + c_3 &= 0 \\ 3c_1 + 6c_2 + c_3 &= 0 \\ c_1 + 4c_2 - 4c_3 &= 0 \end{aligned}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 2 & 3 \\ -2 & -5 & 1 \\ 3 & 6 & 1 \\ 1 & 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the only solution is $c_1 = c_2 = c_3 = 0$.

Thus, the set is linearly independent.

## Linear Independence

**Theorem 4.1.5**

A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ in a vector space $\mathbb{V}$ is linearly dependent if and only if $\vec{v}_i \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\}$ for some $i$, $1 \leq i \leq k$.

**Theorem 4.1.6**

Any set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ in a vector space $\mathbb{V}$ which contains the zero vector is linearly dependent.