Systems of Linear Equations

Definition: An equation in n variables $x_1, \dots, x_n$ that can be written in the form

$$a_1x_1 + \cdots + a_nx_n = b$$

where $a_1, \dots, a_n, b$ are constants is called a linear equation. The constants a_i are called the coefficients of the equation and b is called the right-hand side.

The form $a_1x_1 + \cdots + a_nx_n = b$ is called the standard form of the linear equation.

Systems of Linear Equations

Definition: A set of m linear equations in the same variables $x_1, \dots, x_n$ is called a system of m linear equations in n variables.

A general system of m linear equations in n variables has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Observe that the coefficient a_{ij} represents the coefficient of x_j in the i-th equation.

Definition: A solution to a system of linear equations m in n variables is a vector $\begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$ in $\mathbb{R}^n$ such that all m equations are satisfied when we set $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$. The set of all solutions of a system of linear equations is called the solution set of the system.

Definition: If a system of linear equations has at least one solution, then it is said to be consistent. Otherwise, it is said to be inconsistent.

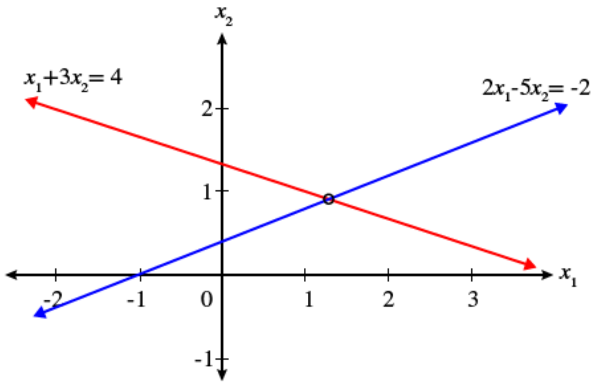
Geometric Interpretation

Example 1

The system of 2 linear equations in 2 variables

$$\begin{aligned} x_1 + 3x_2 &= 4 \\ 2x_1 - 5x_2 &= -2 \end{aligned}$$

graphically represents two lines in $\mathbb{R}^2$.



We see that both lines intersect only at the point (14/11, 10/11).

Hence, the system is consistent with unique solution $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14/11 \\ 10/11 \end{bmatrix}$.

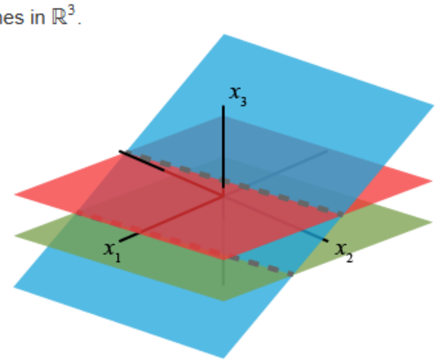
Geometric Interpretation

Example 2

The system of 3 linear equations in 3 variables

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 4 \\ 8x_1 - x_2 + 2x_3 &= 0 \\ -x_1 - 2x_2 + 3x_3 &= 1 \end{aligned}$$

graphically represents three planes in $\mathbb{R}^3$.

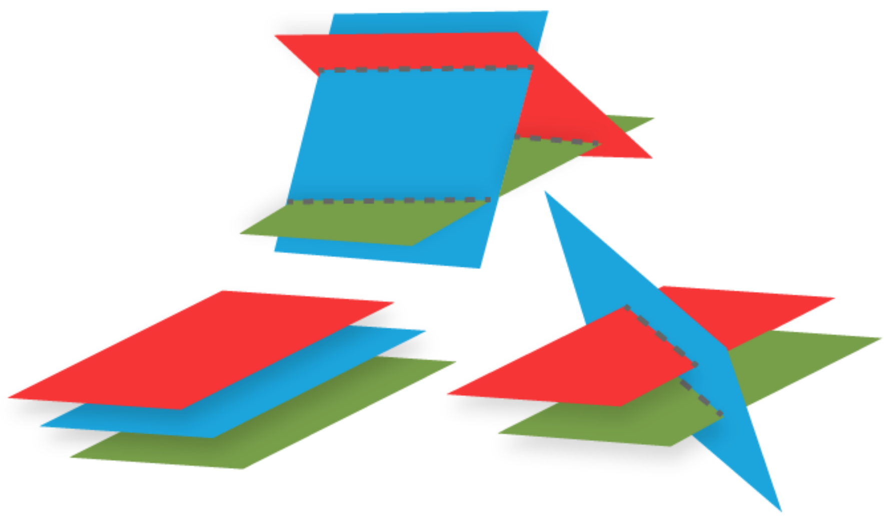


We see that there is no point that lies on all three planes so the system is inconsistent.

Solutions of Systems

The solution set for a system of 3 linear equations in 3 variables can be

Inconsistent



Solutions of Systems

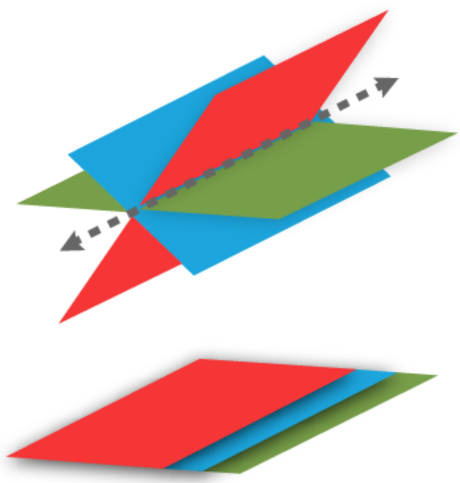
The solution set for a system of 3 linear equations in 3 variables can be

Consistent With A Unique Solution

Solutions of Systems

The solution set for a system of 3 linear equations in 3 variables can be

Consistent With Infinitely Many Solutions



Solutions of Systems

Theorem 2.1.1

If the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

has two distinct solutions $\vec{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$ and $\vec{t} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$, then $\vec{x} = \vec{s} + c(\vec{s} - \vec{t})$ is a distinct solution for each $c \in \mathbb{R}$.

Solutions of Systems

If the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

has two distinct solutions $\vec{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$ and $\vec{t} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$, then $\vec{x} = \vec{s} + c(\vec{s} - \vec{t})$ is a distinct solution for each $c \in \mathbb{R}$.

To prove this theorem we need to prove

1. For any $c \in \mathbb{R}, \vec{x} = \vec{s} + c(\vec{s} - \vec{t})$ is a solution.
2. For every distinct value of c we get a unique solution.

Proof

The i-th equation of the system is

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_i$$

Substituting

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{s} + c(\vec{s} - \vec{t}) = \begin{bmatrix} s_1 + c(s_1 - t_1) \\ \vdots \\ s_n + c(s_n - t_n) \end{bmatrix}$$

into the i-th equation gives

$$\begin{aligned} a_{i1}\big(s_1 + c(s_1 - t_1)\big) + \cdots + a_{in}\big(s_n + c(s_n - t_n)\big) &= b_i \\ a_{i1}s_1 + ca_{i1}s_1 - ca_{i1}t_1 + \cdots + a_{in}s_n + ca_{in}s_n - ca_{in}t_n &= b_i \\ a_{i1}s_1 + \cdots + a_{in}s_n + c\big(a_{i1}s_1 + \cdots + a_{in}s_n\big) - c\big(a_{i1}t_1 + \cdots + a_{in}t_n\big) &= b_i \\ b_i + cb_i - cb_i &= b_i \\ b_i &= b_i \end{aligned}$$

So, the i-th equation is satisfied when $\vec{x} = \vec{s} + c(\vec{s} - \vec{t})$.
Since this is valid for all $1 \le i \le m$, we have shown that $\vec{x} = \vec{s} + c(\vec{s} - \vec{t})$ is a solution for each $c \in \mathbb{R}$.

Solutions of Systems

If the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

has two distinct solutions $\vec{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$ and $\vec{t} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$, then $\vec{x} = \vec{s} + c(\vec{s} - \vec{t})$ is a distinct solution for each $c \in \mathbb{R}$.

To prove this theorem we need to prove

1. For any $c \in \mathbb{R}, \vec{x} = \vec{s} + c(\vec{s} - \vec{t})$ is a solution.
2. For every distinct value of c we get a unique solution.

Proof

Let $c_1, c_2 \in \mathbb{R}$ with $c_1 \neq c_2$ and assume that
$\vec{s} + c_1(\vec{s} - \vec{t}) = \vec{s} + c_2(\vec{s} - \vec{t})$.
This gives

$$\begin{aligned} \vec{s} + c_1(\vec{s} - \vec{t}) &= \vec{s} + c_2(\vec{s} - \vec{t}) \\ c_1(\vec{s} - \vec{t}) &= c_2(\vec{s} - \vec{t}) \\ (c_1 - c_2)(\vec{s} - \vec{t}) &= \vec{0} \end{aligned}$$

Since $c_1 \neq c_2$, this implies that $\vec{s} - \vec{t} = \vec{0}$ and hence $\vec{s} = \vec{t}$.
But this contradicts our assumption that $\vec{s} \neq \vec{t}$. □

This proves that the solution set of a system of m linear equations in n variables must either be empty, contain exactly one vector, or have infinitely many vectors in it.

Solving Systems

If one analyzes the method of solving a system of linear equations by substitution and elimination (see the course notes) one notices a few things.

1. In each step we obtain a new system of linear equations which has the same solution set as the original.
2. It is only the coefficients of the variables that we modify, so we don't actually need to write down the variables each time.
3. We only use two operations: we multiply an equation by a non-zero constant, and we add a multiple of one equation to another.

Matrix Representation of a System

Definition: Two systems of linear equations which have the same solution set are said to be equivalent.

Definition: For a system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

the coefficient matrix is defined to be the rectangular array

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

It is important to note that

1. The i-th row of the coefficient matrix contains the coefficients from the i-th equation in the system.
2. The j-th column of the cofficient matrix contains all the coefficients of x_j in the system.

Matrix Representation of a System

Example

Consider the system of 4 equations in 3 unknowns

$$\begin{aligned} 3x_1 - 2x_2 \qquad &= 4 \\ 5x_1 + x_2 - 8x_3 &= 16 \\ \sqrt{2}x_1 \qquad - 4x_3 &= 0 \\ x_1 + \frac{1}{4}x_2 - x_3 &= -3 \end{aligned}$$

The coefficient matrix of the system is $\begin{bmatrix} 3 & -2 & 0 \\ 5 & 1 & -8 \\ \sqrt{2} & 0 & -4 \\ 1 & 1/4 & -1 \end{bmatrix}$.

The augmented matrix of the system is $\left[\begin{array}{ccc|c} 3 & -2 & 0 & 4 \\ 5 & 1 & -8 & 16 \\ \sqrt{2} & 0 & -4 & 0 \\ 1 & 1/4 & -1 & -3 \end{array}\right]$.

Matrix Representation of a System

Notice that we can write any system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

as a vector equation

$$x_1\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n\begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

The columns of the coefficient matrix of the system are the vectors $\vec{a}_i = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{bmatrix}$ for $1 \leq i \leq n$.

Thus, we can denote the coefficient matrix of the system by $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$.

If we let $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$, then we can denote the augmented matrix of the system by $\left[\begin{array}{ccc|c} \vec{a}_1 & \cdots & \vec{a}_n & \vec{b} \end{array}\right]$ or sometimes simply as $\left[A \mid \vec{b}\right]$.

Matrix Representation of a System

Example

We may write the system of 4 equations in 3 unknowns

$$\begin{aligned} 3x_1 - 2x_2 \qquad &= 4 \\ 5x_1 + x_2 - 8x_3 &= 16 \\ \sqrt{2}x_1 - 4x_3 &= 0 \\ x_1 + \frac{1}{4}x_2 - x_3 &= -3 \end{aligned}$$

as $\left[\begin{array}{ccc|c} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{b} \end{array}\right]$ where

$$\vec{a}_1 = \begin{bmatrix} 3 \\ 5 \\ \sqrt{2} \\ 1 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1/4 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 0 \\ -8 \\ -4 \\ -1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 16 \\ 0 \\ -3 \end{bmatrix}$$