The Identity Matrix and Block Multiplication

Last Lecture

- We defined matrix-matrix multiplication.

In This Lecture

- We will look at the multiplicative identity.
- We will look at how matrix-matrix multiplication can be split into smaller blocks.

The Identity Matrix

We want to find the multiplicative identity of matrix-matrix multiplication.
That is, we want to find a matrix I such that $AI = A = IA$.
Let A be an $m \times n$ matrix.
For $AI = A$ to be possible, I must be an $n \times n$ matrix.
On the other hand, if $IA = A$, then I must be $m \times m$.
Hence, $M_{m \times n}(\mathbb{R})$ does not have a multiplicative identity whenever $m \neq n$.
So, we now just try to find the multiplicative identity for $M_{n \times n}(\mathbb{R})$.

The Identity Matrix

If $I = \begin{bmatrix} \vec{i}_1 & \cdots & \vec{i}_n \end{bmatrix}$ is the multiplicative identity for $M_{n\times n}(\mathbb{R})$ and $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \in M_{n\times n}(\mathbb{R})$, then we have $A = AI = IA$.

Expanding $A = AI$ we get

$$\begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} = A\begin{bmatrix} \vec{i}_1 & \cdots & \vec{i}_n \end{bmatrix} = \begin{bmatrix} A\vec{i}_1 & \cdots & A\vec{i}_n \end{bmatrix}$$

Hence, $\vec{a}_j = A\vec{i}_j$ for $1 \leq j \leq n$.

Recall that the j-th standard basis vector $\vec{e}_j$ satisfies $a_j = A\vec{e}_j$ for $1 \leq j \leq n$.

Hence, taking $I = \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{bmatrix}$ gives the following theorem.

Theorem 3.1.5

If $I = \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{bmatrix}$, then $AI = A = IA$ for any $n \times n$ matrix A.

The Identity Matrix

Theorem 3.1.6

The multiplicative identity for $M_{n\times n}(\mathbb{R})$ is unique.

Proof

Assume that I_1 and I_2 are both multiplicative identities for $M_{n\times n}(\mathbb{R})$.

Since I_1 is a multiplicative identity, it satisfies $A = I_1A$ for any $A \in M_{n\times n}(\mathbb{R})$.

Since $I_2 \in M_{n\times n}(\mathbb{R})$, we have that $I_2 = I_1I_2$.

But I_2 is also a multiplicative identity, so it satisfies $BI_2 = B$ for any $B \in M_{n\times n}(\mathbb{R})$

Therefore $I_2 = I_1I_2 = I_1$.

The Identity Matrix

Definition: The $n \times n$ matrix I such that $(I)_{ii} = 1$ for $1 \leq i \leq n$, and $(I)_{ij} = 0$ whenever $i \neq j$ is called the identity matrix.

We sometimes will denote the identity matrix as I_n to stress the size of the matrix.

Example

The 2×2 identity matrix is

$$I_2 = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The 4×4 identity matrix is

$$I_4 = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Block Matrices

We now look at subdividing a matrix into different sized submatrices called blocks.

Definition: If A is an $m \times n$ matrix, then we can write A as the $k \times \ell$ block matrix

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1\ell} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{k\ell} \end{bmatrix}$$

where A_{ij} is a block such that all blocks in the i-th row have the same number of rows and all blocks in the j-th column have the same number of columns.

Block Matrices

Example

Let $A = \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 2 \end{bmatrix}$.

Block Matrices

Example

Let $A = \left[\begin{array}{cc|cc} 1 & -1 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 2 \end{array}\right]$.

We can write A as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where $A_{11} = \begin{bmatrix} 1 & -1 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 3 & 4 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$, and $A_{22} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$.

Block Matrices

Example

Let $A = \left[\begin{array}{c|cc|c} 1 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 3 & 1 & 2 \end{array}\right]$.

We can write A as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where $A_{11} = \begin{bmatrix} 1 & -1 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 3 & 4 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$, and $A_{22} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$.

Or, we could write A as

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

where $A_{11} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A_{12} = \begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix}$, $A_{13} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 0 \end{bmatrix}$, $A_{22} = \begin{bmatrix} 3 & 1 \end{bmatrix}$, and $A_{23} = \begin{bmatrix} 2 \end{bmatrix}$.

Block Multiplication

We actually defined matrix-matrix multiplication in terms of blocks.
We wrote the second matrix B as block matrix $B = [B_1 \cdots B_n]$ where each block is a column of B and we wrote the first matrix A as a single block.
We then got $AB = \begin{bmatrix} AB_1 \cdots AB_n \end{bmatrix}$.

Example

If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}$, then

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} & A_{11}B_{13} + A_{12}B_{13} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} & A_{21}B_{13} + A_{22}B_{32} \end{bmatrix}$$

assuming that we have divided A and B into blocks such that all of the required products of the blocks is defined.