Addition and Scalar Multiplication of Polynomials

Definition: If $p(x) = a_0 + a_1x + \cdots + a_nx^n$ and $q(x) = b_0 + b_1x + \cdots + b_nx^n$ are both polynomials of degree less than or equal to n, and if t is a scalar (that is, $t \in \mathbb{R}$), then there are polynomials $(p+q)(x)$ and $(tp)(x)$ of degree less than or equal to n defined as follows:

$$(p+q)(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

and

$$(tp)(x) = (ta_0) + (ta_1)x + \cdots + (ta_n)x^n$$

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Examples

$$3(2 + 5x - 4x^2) = ((3)(2)) + ((3)(5))x + ((3)(-4))x^2 = 6 + 15x - 12x^2$$

When adding polynomials, line up the corresponding entries in columns:

$$(1 + 2x + 6x^2 + 5x^3) + (3 - x - 9x^2 + 3x^3) = \begin{array}{cccc} 1 & +2x & +6x^2 & +5x^3 \\ 3 & -x & -9x^2 & +3x^3 \\ \hline 4 & +x & -3x^2 & +8x^3 \end{array}$$

This is particularly useful when some of the coefficients are zero:

$$(3 - x^2 + 8x^4) + (2 + 8x - 4x^2 + 7x^3 - 5x^5) = \begin{array}{cccccc} 3 & & -x^2 & & +8x^4 & \\ 2 & +8x & -4x^2 & +7x^3 & & -5x^5 \\ \hline 5 & +8x & -5x^2 & +7x^3 & +8x^4 & -5x^5 \end{array}$$

When scalar multiplication and addition are combined, distribute the scalar first, and then line it up in columns to add:

$$6(1 - 3x - 5x^2) - 2(9 - x^2) = (6 - 18x - 30x^2) + (-18 + 2x^2) = \begin{array}{ccc} 6 & -18x & -30x^2 \\ -18 & & +2x^2 \\ \hline -12 & -18x & -28x^2 \end{array}$$

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Theorem 4.1.1

Let $p(x)$, $q(x)$, and $r(x)$ be polynomials of degree at most n and let $s, t \in \mathbb{R}$. Then

1. $p(x) + q(x)$ is a polynomial of degree at most n — closed under addition
2. $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$ — addition is associative
3. The polynomial $0 = 0 + 0x + \cdots + 0x^n$ (called the zero polynomial) satisfies $p(x) + 0 = p(x) = 0 + p(x)$ for any polynomial $p(x)$ — additive identity
4. For each polynomial $p(x)$, there exists an additive inverse, denoted $(-p)(x)$, with the property that $p(x) + (-p)(x) = 0$. In particular, $(-p)(x) = -1p(x)$ — additive inverse
5. $p(x) + q(x) = q(x) + p(x)$ — addition is commutative
6. $sp(x)$ is a polynomial of degree at most n — closed under scalar multiplication
7. $s(tp(x)) = (st)p(x)$ — scalar multiplication is associative
8. $(s + t)p(x) = sp(x) + tp(x)$ — scalar addition is distributive
9. $s(p(x) + q(x)) = sp(x) + sq(x)$ — scalar multiplication is distributive
10. $1p(x) = p(x)$ — scalar multiplicative identity