Change of Coordinates and Linear Mappings

Example

Let $L : M(2,2) \to P_2$ be defined by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b) + (a+c)x + (a+d)x^2$$

We found the matrix for L with respect to two different pairs of bases. If $\mathcal{B}_1$ is the standard basis for $M(2,2)$ and $\mathcal{C}_1$ is the standard basis for P_2, we have

$$_{\mathcal{C}_1}[L]_{\mathcal{B}_1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

If $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ and $\mathcal{C}_2 = \{1, 1+x, 1+x+x^2\}$, we have

$$_{\mathcal{C}_2}[L]_{\mathcal{B}_2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

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Since it is reasonably easy to find the coordinates for a matrix with respect to the standard basis, we can easily use our first matrix to compute $L(\mathbf{x})$.

Before we can use the second matrix to compute $L(\mathbf{x})$, we would first need to find the $\mathcal{B}_2$-components of $\mathbf{x}$. Now, if we only wanted to do that for one vector $\mathbf{x}$, we would do so by setting up an equation and row reducing a matrix.

However, the more likely situation is that we would want to compute $L(\mathbf{x})$ for many (perhaps even all) matrices $\mathbf{x} \in M(2,2)$, and the faster way to do that would be to find the change of coordinates matrix from the standard basis ($\mathcal{B}_1$) to $\mathcal{B}_2$.
We've already looked at how to do that, so the point now is that we can also use our change of coordinates matrix to find $_{\mathcal{C}_2}[L]_{\mathcal{B}_2}$ if we already have $_{\mathcal{C}_1}[L]_{\mathcal{B}_1}$.

Let R be the change of coordinates matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$, and let Q be the change of coordinates matrix from $\mathcal{C}_1$ to $\mathcal{C}_2$. Then we have that

$$[L(\mathbf{x})]_{\mathcal{C}_2} = Q[L(\mathbf{x})]_{\mathcal{C}_1} = Q_{\mathcal{C}_1}[L]_{\mathcal{B}_1}[\mathbf{x}]_{\mathcal{B}_1}$$

$$[L(\mathbf{x})]_{\mathcal{C}_2} = {}_{\mathcal{C}_2}[L]_{\mathcal{B}_2}[\mathbf{x}]_{\mathcal{B}_2} = {}_{\mathcal{C}_2}[L]_{\mathcal{B}_2}R[\mathbf{x}]_{\mathcal{B}_1}$$

So, for every vector $[\mathbf{x}]_{\mathcal{B}_1}$ in $\mathbb{R}^n$, which will in fact be every vector in $\mathbb{R}^n$, we have that

$$Q_{\mathcal{C}_1}[L]_{\mathcal{B}_1}[\mathbf{x}]_{\mathcal{B}_1} = {}_{\mathcal{C}_2}[L]_{\mathcal{B}_2}R[\mathbf{x}]_{\mathcal{B}_1}$$

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It's been a while, but if you recall Theorem 3.1.4, then you see that this means

$$Q_{C_1}[L]_{B_1} = {}_{C_2}[L]_{B_2}R$$

Since R is a change of coordinates matrix, R is invertible (as R^{-1} is the change of coordinates matrix from B_2 to B_1) and so we have

$${}_{C_2}[L]_{B_2} = Q_{C_1}[L]_{B_1}R^{-1}$$

Now, I started this explanation in regards to our example, but I finished using only general notation, so, in fact, we've shown that this holds in general.

Summary

Let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping, B_1 and B_2 be bases for $\mathbb{V}$, and C_1 and C_2 be bases for $\mathbb{W}$.
Then, if we let R be the change of coordinates matrix from B_1 to B_2 and Q be the change of coordinates matrix from C_1 to C_2, we have

$${}_{C_2}[L]_{B_2} = Q_{C_1}[L]_{B_1}R^{-1}$$

Change of Coordinates and Linear Mappings

Example

Let's go ahead and apply this result to our example. To do so, we will first need to find R^{-1} and Q. We have

$$\begin{aligned} R^{-1} &= \left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{B_1} \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}_{B_1} \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}_{B_1} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{B_1} \right] \\ &= \left[\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right] \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} Q &= \left[[1]_{C_2} \quad [x]_{C_2} \quad [x^2]_{C_2} \right] \\ &= \left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right] \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

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Now it is a simple case of matrix multiplication:

$$\begin{aligned} {}_{C_2}[L]_{B_2} &= Q_{C_1}[L]_{B_1}R^{-1} \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix} \end{aligned}$$

Change of Coordinates and Linear Mappings

Course Author's Note

I have deviated from the text again here, in order to provide the most general result.
So let's see how to take our result, and make it the same as the result in the text.

For starters, the text only looks at the case where L is a linear operator, so $\mathbb{V}$ and $\mathbb{W}$ are the same.
It further makes the restriction that $\mathbb{V}$ is $\mathbb{R}^n$, although that is rather extreme.
Why does the text bother with the notation $[\vec{x}]_S$, when $[\vec{x}]_S = \vec{x}$?
There is value in assuming that one of our bases is the standard basis S.

The other significant restriction is that the book sets $\mathcal{B}_1 = \mathcal{C}_1$ and $\mathcal{B}_2 = \mathcal{C}_2$.
That is, not only does it have the domain and codomain equal to each other, but we are looking at the matrix with respect to the same basis for the domain and codomain.

So, to get the result in the book, you have to set $\mathcal{B}_1 = \mathcal{C}_1 = S$ and $\mathcal{B}_2 = \mathcal{C}_2 = \mathcal{B}$.
Then R is the change of coordinates matrix from S to $\mathcal{B}$, and Q is also the change of coordinates matrix from S to $\mathcal{B}$. As such, $R = Q$, and our result is now

$$[L]_{\mathcal{B}} = R[L]_S R^{-1}$$

This almost what the text writes, except we need to replace R with P^{-1} and R^{-1} with P.
And that is because they are defined as inverses.
While I found it more straightforward to use the change of coordinates matrix from S to $\mathcal{B}$, the text based its result on the change of coordinates matrix from $\mathcal{B}$ to S.

Change of Coordinates and Linear Mappings

Example

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $L(a,b,c) = (2a, a+b, 4b+c)$, and let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

To find $[L]_\mathcal{B}$, we will first find $[L]_\mathcal{S}$:

$$\begin{aligned} [L]_\mathcal{S} &= \left[L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \quad L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \quad L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \right] \\ &= \left[\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \end{aligned}$$

Change of Coordinates and Linear Mappings

Example

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $L(a,b,c) = (2a, a+b, 4b+c)$, and let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Next, we need to find P (the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{S}$):

$$\begin{aligned} P &= \left[\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_\mathcal{S} \quad \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}_\mathcal{S} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}_\mathcal{S} \right] \\ &= \left[\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right] \\ &= \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{aligned}$$

Change of Coordinates and Linear Mappings

Example

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $L(a,b,c) = (2a, a+b, 4b+c)$, and let $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Now we need to find P^{-1}, using the matrix inverse algorithm:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] R_3 - R_1 \sim \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right] R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right] (1/2)R_3 \sim \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right] \begin{array}{l} R_1 + R_3 \\ R_2 - R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right] R_1 - 2R_2 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & -1/2 & 3/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

So $P^{-1} = \begin{bmatrix} -1/2 & -1/2 & 3/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}$.

Change of Coordinates and Linear Mappings

Example

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $L(a,b,c) = (2a, a+b, 4b+c)$, and let $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

At long last, we can compute $[L]_B$:

$$\begin{aligned} [L]_B &= P^{-1}[L]_S P \\ &= \begin{bmatrix} -1/2 & -1/2 & 3/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1/2 & -1/2 & 3/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 1 & 3 & 0 \\ 1 & 5 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 4 & 7 \\ 1 & 1 & -3 \\ 0 & 2 & 3 \end{bmatrix} \end{aligned}$$

This is, of course, the same matrix we found in the previous lecture.