Dimension

Lemma 4.3.2

Suppose that $\mathbb{V}$ is a vector space and $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \mathbb{V}$. If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a linearly independent set in $\mathbb{V}$, then $k \leq n$. In words, this says that the number of vectors in a linearly independent set will be less than, or equal to, the number of vectors in a spanning set.

Theorem 4.3.3

If $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\mathcal{C} = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ are both bases of a vector space $\mathbb{V}$, then $k = n$.

Note that this theorem does not say the a vector space will have a unique basis, merely that the number of vectors in a basis is unique.

We call this unique number of basis vectors the **dimension** of the vector space.

Dimension

Definition: If a vector space $\mathbb{V}$ has a basis with n vectors, then we say that the dimension of $\mathbb{V}$ is n and write

$$\dim \mathbb{V} = n$$

The dimension of the trivial vector space $\mathbb{O} = \{\mathbf{0}\}$ is defined to be 0, consistent with our definition that the basis for $\mathbb{O}$ is the empty set.
If a vector space $\mathbb{V}$ does not have a basis with finitely many elements, then $\mathbb{V}$ is called **infinite-dimensional**.

We will not be studying the properties of infinite-dimensional spaces in this course.

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Dimension

Examples

(1) $\mathbb{R}^n$ has standard basis $\{\vec{e}_1, \dots, \vec{e}_n\}$, so $\dim \mathbb{R}^n = n$.

(2) $M(m,n)$ has a standard basis consisting of the mn matrices with a 1 in one entry and a 0 in the other entries. As such, $\dim M(m,n) = mn$.

(3) P_n has standard basis $\{1, x, x^2, \dots, x^n\}$, so $\dim P_n = n + 1$ (count carefully!)

(4) $\mathcal{F}$, $\mathcal{F}(a,b)$, and $C(a,b)$ are all infinite-dimensional spaces.

Dimension

Example

In the previous lecture, we found that

$$\mathcal{T}_2 = \left\{ \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -3 & 7 \\ 11 & -5 \end{bmatrix}, \begin{bmatrix} 2 & -8 \\ -12 & 7 \end{bmatrix} \right\}$$

is a basis for $\operatorname{Span} \mathcal{T}$, where

$$\mathcal{T} = \left\{ \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -3 & 7 \\ 11 & -5 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ 7 & -5 \end{bmatrix}, \begin{bmatrix} 2 & -8 \\ -12 & 7 \end{bmatrix}, \begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix} \right\}$$

Since $\mathcal{T}_2$ contains 3 matrices, this means that the dimension of $\operatorname{Span} \mathcal{T}$ is 3.

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