

Linear Mappings

Let's return now to a concept we developed in our study of \mathbb{R}^n : linear mappings. As with bases, linear mappings will be of much greater use in the general world of vector spaces than they were in simply the \mathbb{R}^n spaces. We will, of course, start with a definition.

Definition: If V and W are vector spaces over \mathbb{R} , a function $L : V \rightarrow W$ is a **linear mapping** if it satisfies the linearity properties

L1. $L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y})$

L2. $L(t\mathbf{x}) = tL(\mathbf{x})$

for all $\mathbf{x}, \mathbf{y} \in V$ and $t \in \mathbb{R}$. If $W = V$, then L may be called a **linear operator**.

Note: These two properties can be combined into one statement: $L(t\mathbf{x} + \mathbf{y}) = tL(\mathbf{x}) + L(\mathbf{y})$.

Linear Mappings

Example

The mapping $L : M(2, 2) \rightarrow P_2$ defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + b) + (a + c)x + (a + d)x^2$ is a linear mapping, because

$$\begin{aligned} L(t\mathbf{x} + \mathbf{y}) &= L\left(t\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \\ &= L\left(\begin{bmatrix} ta_1 + a_2 & tb_1 + b_2 \\ tc_1 + c_2 & td_1 + d_2 \end{bmatrix}\right) \\ &= (ta_1 + a_2 + tb_1 + b_2) + (ta_1 + a_2 + tc_1 + c_2)x + (ta_1 + a_2 + td_1 + d_2)x^2 \\ &= (ta_1 + tb_1) + (ta_1 + tc_1)x + (ta_1 + td_1)x^2 + (a_2 + b_2) + (a_2 + c_2)x + (a_2 + d_2)x^2 \\ &= t((a_1 + b_1) + (a_1 + c_1)x + (a_1 + d_1)x^2) + ((a_2 + b_2) + (a_2 + c_2)x + (a_2 + d_2)x^2) \\ &= tL\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \\ &= tL(\mathbf{x}) + L(\mathbf{y}) \end{aligned}$$

Linear Mappings

Example

The mapping $M : P_3 \rightarrow P_3$ defined by $M(a + bx + cx^2 + dx^3) = a^2 + abx + acx^2 + adx^3$ is not linear. Consider, for example, that

$$M(1 + 2x + x^2 + 2x^3) = 1 + 2x + x^2 + 2x^3 \text{ and } M(4 - x + 2x^2 - x^3) = 16 - 4x + 8x^2 - 4x^3$$

and so we have that

$$\begin{aligned} M(1 + 2x + x^2 + 2x^3) + M(4 - x + 2x^2 - x^3) &= (1 + 2x + x^2 + 2x^3) + (16 - 4x + 8x^2 - 4x^3) \\ &= 17 - 2x + 9x^2 - 2x^3 \end{aligned}$$

but that

$$\begin{aligned} M((1 + 2x + x^2 + 2x^3) + (4 - x + 2x^2 - x^3)) &= M(5 + x + 3x^2 + x^3) \\ &= 25 + 5x + 15x^2 + 5x^3 \end{aligned}$$

and so we have that

$M(1 + 2x + x^2 + 2x^3) + M(4 - x + 2x^2 - x^3) \neq M((1 + 2x + x^2 + 2x^3) + (4 - x + 2x^2 - x^3))$. This means that M is not closed under addition, so it is not a linear mapping.

Linear Mappings

It happens that M is also not closed under scalar multiplication.

Consider, for example, that $M(1 + 2x + x^2 + 2x^3) = 1 + 2x + x^2 + 2x^3$,

$$\text{so } 5M(1 + 2x + x^2 + 2x^3) = 5(1 + 2x + x^2 + 2x^3) = 5 + 10x + 5x^2 + 10x^3,$$

$$\text{but } M(5(1 + 2x + x^2 + 2x^3)) = M(5 + 10x + 5x^2 + 10x^3) = 25 + 50x + 25x^2 + 50x^3,$$

$$\text{so } 5M(1 + 2x + x^2 + 2x^3) \neq M(5(1 + 2x + x^2 + 2x^3)).$$

Note:

1. If you only perform linear combinations (addition, multiplication by constant), then it is probably linear.
2. If your mapping involves multiplication, roots, or exponents, then it is probably not linear.