Rowspace, Columnspace, and Nullspace

Once one has defined a linear mapping L from $\mathbb{C}^n$ to $\mathbb{C}^m$, and found it's $m \times n$ matrix $[L] = A$, we can find the following subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$.

Definition: The nullspace of an $m \times n$ matrix A is

$$\text{Null}(A) = \{\vec{z} \in \mathbb{C}^n \,|A\vec{z} = \vec{0}\}$$

Definition: Let A be an $m \times n$ matrix, and let $\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_n \in \mathbb{C}^m$ be the columns of A. Then the columnspace of A, written $\text{Col}(A)$, is

$$\text{Span}\{\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_n\}$$

Recall that the columnspace of A is the same as the range of the matrix mapping $A\vec{z}$, or the corresponding linear mapping $L(\vec{z})$.

Definition: Given an $m \times n$ matrix A, the rowspace of A is the subspace spanned by the rows of A (regarded as vectors) and is denoted $\text{Row}(A)$.

We can find a basis for each of these subspaces the same as we did in $\mathbb{R}^n$.

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Example

To find a basis for the rowspace, columnspace, and nullspace of

$$A = \begin{bmatrix} 1 & i & 2+3i & 1 & 1+2i \\ -2i & 1+i & 3-i & 1-i & 2 \\ -1+i & 2 & 4+2i & -2i & i \end{bmatrix}$$

we first need to find the reduced row echelon form of A.

$$\begin{bmatrix} 1 & i & 2+3i & 1 & 1+2i \\ -2i & 1+i & 3-i & 1-i & 2 \\ -1+i & 2 & 4+2i & -2i & i \end{bmatrix} \begin{matrix} \\ R_2 + 2iR_1 \\ R_3 + (1-i)R_1 \end{matrix} \sim \begin{bmatrix} 1 & i & 2+3i & 1 & 1+2i \\ 0 & -1+i & -3+3i & 1+i & -2+2i \\ 0 & 3+i & 9+3i & 1-3i & 3+2i \end{bmatrix} \frac{(-1-i)}{2} R_2$$

$$\sim \begin{bmatrix} 1 & i & 2+3i & 1 & 1+2i \\ 0 & 1 & 3 & -i & 2 \\ 0 & 3+i & 9+3i & 1-3i & 3+2i \end{bmatrix} R_3 + (-3-i)R_2 \sim \begin{bmatrix} 1 & i & 2+3i & 1 & 1+2i \\ 0 & 1 & 3 & -i & 2 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \frac{1}{3} R_3$$

$$\sim \begin{bmatrix} 1 & i & 2+3i & 1 & 1+2i \\ 0 & 1 & 3 & -i & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 + (-1-2i)R_3 \\ R_2 - 2R_3 \\ \end{matrix} \sim \begin{bmatrix} 1 & i & 2+3i & 1 & 0 \\ 0 & 1 & 3 & -i & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} R_1 - iR_2 \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & -i & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

To find a basis for the rowspace of A, we first need to recall that, for any matrix B that is row equivalent to A, the rowspace of B is the same as the rowspace of A.

In particular, the rowspace of the reduced row echelon form of A is the same as the rowspace of A. And the structure of any reduced row eceholon form matrix ensures that the non-zero rows are linearly independent. (Just look at the entries that correspond to the leading 1; only one row has a non-zero entry.)

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Since our particular RREF does not have any rows of all zeros, we find that the rows of the RREF of A form a basis for the rowspace of A: $\left\{\begin{bmatrix}1\\0\\2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\3\\-i\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$.

To find a basis for the columnspace of A, we need to determine which columns of A are linearly independent, and which are dependent members. If we think of A as the coefficient matrix for the equation

$$\alpha_1\begin{bmatrix}1\\-2i\\-1+i\end{bmatrix} + \alpha_2\begin{bmatrix}i\\1+i\\2\end{bmatrix} + \alpha_3\begin{bmatrix}2+3i\\3-i\\4+2i\end{bmatrix} + \alpha_4\begin{bmatrix}1\\1-i\\-2i\end{bmatrix} + \alpha_5\begin{bmatrix}1+2i\\2\\i\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

then the RREF of A tells us that

$$\begin{aligned}\alpha_1 + 2\alpha_3 &= 0\\ \alpha_2 + 3\alpha_3 - i\alpha_4 &= 0\\ \alpha_5 &= 0\end{aligned}$$

If we replace the variable α_3 with the parameter β and the variable α_4 with the parameter γ, then we see that the general solution to this equation is

$$\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\\\alpha_5\end{bmatrix} = \beta\begin{bmatrix}-2\\-3\\1\\0\\0\end{bmatrix} + \gamma\begin{bmatrix}0\\i\\0\\1\\0\end{bmatrix}$$

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If we plug in $\beta = 1$ and $\gamma = 0$, we see that $\begin{bmatrix}-2\\-3\\1\\0\\0\end{bmatrix}$ is a solution, which means that

$$-2\begin{bmatrix}1\\-2i\\-1+i\end{bmatrix} - 3\begin{bmatrix}i\\1+i\\2\end{bmatrix} + \begin{bmatrix}2+3i\\3-i\\4+2i\end{bmatrix} + 0\begin{bmatrix}1\\1-i\\-2i\end{bmatrix} + 0\begin{bmatrix}1+2i\\2\\i\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

or that

$$\begin{bmatrix}2+3i\\3-i\\4+2i\end{bmatrix} = 2\begin{bmatrix}1\\-2i\\-1+i\end{bmatrix} + 3\begin{bmatrix}i\\1+i\\2\end{bmatrix}$$

As such, $\begin{bmatrix}2+3i\\3-i\\4+2i\end{bmatrix}$ is a dependent member. By plugging in $\beta = 0$ and $\gamma = 1$, we get that $\begin{bmatrix}0\\i\\0\\1\\0\end{bmatrix}$ is a solution to our equation, and thus that

$$\begin{bmatrix}1\\1-i\\-2i\end{bmatrix} = -i\begin{bmatrix}i\\1+i\\2\end{bmatrix}$$

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And so $\begin{bmatrix} 1 \\ 1-i \\ -2i \end{bmatrix}$ is also a dependent member. Thus, we know that

$$\text{Span}\left\{\begin{bmatrix} 1 \\ -2i \\ -1+i \end{bmatrix}, \begin{bmatrix} i \\ 1+i \\ 2 \end{bmatrix}, \begin{bmatrix} 2+3i \\ 3-i \\ 4+2i \end{bmatrix}, \begin{bmatrix} 1 \\ 1-i \\ -2i \end{bmatrix}, \begin{bmatrix} 1+2i \\ 2 \\ i \end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix} 1 \\ -2i \\ -1+i \end{bmatrix}, \begin{bmatrix} i \\ 1+i \\ 2 \end{bmatrix}, \begin{bmatrix} 1+2i \\ 2 \\ i \end{bmatrix}\right\}$$

To see that this latter set is linearly independent, we would row reduce the matrix

$$\begin{bmatrix} 1 & i & 1+2i \\ -2i & 1+i & 2 \\ -1+i & 2 & i \end{bmatrix}$$

using the exact same steps as before, and we would end up with the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the rank of this matrix is the same as the number of columns, our set is linearly independent. As such, we see that the columns of A that correspond to the columns of the reduced row echelon form that have a leading 1 form a basis for the columnspace of A:

$$\left\{\begin{bmatrix} 1 \\ -2i \\ -1+i \end{bmatrix}, \begin{bmatrix} i \\ 1+i \\ 2 \end{bmatrix}, \begin{bmatrix} 1+2i \\ 2 \\ i \end{bmatrix}\right\}$$

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Example

To find a basis for the nullspace, we need to find the general solution to the equation $A\vec{z} = \vec{0}$. Luckily, we have already done this. In our efforts to find a basis for the columnspace, we found that the general solution to $A\vec{z} = \vec{0}$ is

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \beta\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \gamma\begin{bmatrix} 0 \\ i \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

This means that

$$\left\{\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

is a spanning set for the nullspace, and as these vectors are clearly not multiples of each other (even in the complex numbers), this set is linearly independent. So

$$\left\{\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

is a basis for the nullspace.

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In this example, I have reviewed the theory behind finding these bases as well as the technique. You will not need to restate this entire process on the assignments or the final exam for these problems though.

Simply finding the reduced row echelon form and obtaining the correct basis from the RREF matrix will suffice. On the other hand, if you need more of a refresher on these techniques, refer back to section 3.4 in the text.

In general, much of what we learned about real vector spaces will translate directly into complex vector spaces, since much of it depends on the fundamental properties of vector spaces.

Instead of continuing to focus on all the ways that complex vector spaces are the same as real vector spaces, we instead will turn our attention to the ways in which they are different, perhaps taking for granted a few similarities not yet stated along the way.