

## The Matrix of a Linear Mapping

### Previously

- When we first studied linear mappings between  $\mathbb{R}^n$  and  $\mathbb{R}^m$ , we were always able to find a matrix  $A$  such that our linear mapping  $L(\vec{x})$  was the same as  $A\vec{x}$ .
- That's because, while  $\vec{x} \in \mathbb{R}^n$  isn't technically a matrix, we could temporarily think of  $\vec{x}$  like a matrix, and the matrix product  $A\vec{x}$  had the properties we wanted.

We would like to be able to do this for any linear mapping, not just ones not just ones between  $\mathbb{R}^n$  and  $\mathbb{R}^m$ .

But the product  $A(2 + 3x - x^2)$  (for example) doesn't make any sense at all!

## The Matrix of a Linear Mapping

### Previously

So before we can find a matrix for a linear mapping whose domain is not  $\mathbb{R}^n$ , we first need to find some way to make our random vector look like a vector from  $\mathbb{R}^n$ .

Once we fix a basis  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  for a vector space  $\mathbb{V}$ , we can look at the  $\mathcal{B}$ -coordinates for  $\mathbf{x}$ .

That's only the first step in finding the matrix of a general linear mapping  $L : \mathbb{V} \rightarrow \mathbb{W}$  though. Because our product  $A[\mathbf{x}]_{\mathcal{B}}$  will be a vector from  $\mathbb{R}^m$ , not  $\mathbb{W}$ .

So we will also need to fix a basis  $\mathcal{C} = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$  for  $\mathbb{W}$ , and we will have our output be in  $\mathcal{C}$ -coordinates. So while we can't find a matrix for  $L$  directly, we can find a matrix for  $L$  relative to the  $\mathcal{B}$  and  $\mathcal{C}$  coordinates.

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As such, instead of simply denoting the matrix for the linear mapping as  $[L]$ , we denote it as  ${}_C[L]_B$ , so that we remember what coordinates we are using. And this matrix  ${}_C[L]_B$  will be such that

$$[L(\mathbf{x})]_C = {}_C[L]_B[\mathbf{x}]_B$$

We know that the matrix  ${}_C[L]_B$  must exist, since the equation above defines a linear mapping from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ .

The first thing we want to note is that there is a vector  $[\mathbf{x}]_B = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$  such that  $\mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$ .

Then we have that

$$\begin{aligned} L(\mathbf{x}) &= L(x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n) \\ &= x_1L(\mathbf{v}_1) + \cdots + x_nL(\mathbf{v}_n) \end{aligned}$$

But, recalling Theorem 4.4.1, we then have that

$$\begin{aligned} [L(\mathbf{x})]_C &= [x_1L(\mathbf{v}_1) + \cdots + x_nL(\mathbf{v}_n)]_C \\ &= x_1[L(\mathbf{v}_1)]_C + \cdots + x_n[L(\mathbf{v}_n)]_C \\ &= \begin{bmatrix} [L(\mathbf{v}_1)]_C & \cdots & [L(\mathbf{v}_n)]_C \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= [{}_C[L]_B][\mathbf{x}]_B \end{aligned}$$

From this, we see that

$${}_C[L]_B = \begin{bmatrix} [L(\mathbf{v}_1)]_C & \cdots & [L(\mathbf{v}_n)]_C \end{bmatrix}$$

### The Matrix of a Linear Mapping

**Definition:** Let  $V$  be a vector space with basis  $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ , let  $W$  be a vector space with basis  $C$ , and let  $L : V \rightarrow W$  be a linear mapping. We define the **matrix of  $L$  with respect to the bases  $B$  and  $C$**  to be the matrix

$${}_C[L]_B = \begin{bmatrix} [L(\mathbf{v}_1)]_C & \cdots & [L(\mathbf{v}_n)]_C \end{bmatrix}$$

### The Matrix of a Linear Mapping

#### Example

Let  $L : M(2, 2) \rightarrow P_2$  be defined by  $L \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a + b) + (a + c)x + (a + d)x^2$ , and

let  $\mathcal{B}$  be the standard basis for  $M(2, 2)$ , that is,  $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ . Let  $\mathcal{C}$  be the standard basis for  $P_2$ , that is,  $\mathcal{C} = \{1, x, x^2\}$ .

To find  ${}_c[L]_{\mathcal{B}}$ , we need to compute the  $\mathcal{C}$ -coordinates of the image under  $L$  of the  $\mathcal{B}$  basis vectors.

$$\begin{aligned} L \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) &= 1 + x + x^2, & \text{so } \left[ L \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \right]_{\mathcal{C}} &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ L \left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) &= 1, & \text{so } \left[ L \left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \right]_{\mathcal{C}} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ L \left( \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) &= x, & \text{so } \left[ L \left( \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) \right]_{\mathcal{C}} &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ L \left( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) &= x^2, & \text{so } \left[ L \left( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \right]_{\mathcal{C}} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

And thus we have that  ${}_c[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

### The Matrix of a Linear Mapping

#### Example

Let  $L : M(2, 2) \rightarrow P_2$  be defined by  $L \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a + b) + (a + c)x + (a + d)x^2$ , and let

$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$  and  $\mathcal{C} = \{1, 1 + x, 1 + x + x^2\}$ .

To find  ${}_c[L]_{\mathcal{B}}$ , we need to compute the  $\mathcal{C}$ -coordinates of the image under  $L$  of the  $\mathcal{B}$  basis vectors.

$$\begin{aligned} L \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) &= 1 + x + x^2, & \text{so } \left[ L \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \right]_{\mathcal{C}} &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\ L \left( \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right) &= 2 + x + x^2, & \text{so } \left[ L \left( \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right) \right]_{\mathcal{C}} &= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\ L \left( \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right) &= 2 + 2x + x^2, & \text{so } \left[ L \left( \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right) \right]_{\mathcal{C}} &= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ L \left( \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) &= 2 + 2x + 2x^2, & \text{so } \left[ L \left( \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) \right]_{\mathcal{C}} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \end{aligned}$$

And thus we have that  ${}_c[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

If our domain and codomain are the same vector space, then we might use the same basis  $\mathcal{B}$  for both.

In these situations, we simply write  ${}_c[L]_{\mathcal{B}}$  as  $[L]_{\mathcal{B}}$ .

## The Matrix of a Linear Mapping

### Example

Let  $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be defined by  $L(a, b, c) = (2a, a + b, 4b + c)$ , and let  $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$  and  $\mathcal{C} = \{1, 1 + x, 1 + x + x^2\}$ .

To find  $[L]_{\mathcal{B}}$ , we need to compute the  $\mathcal{B}$ -coordinates of the image under  $L$  of the  $\mathcal{B}$  basis vectors.

First, let's just find the image under  $L$  of the  $\mathcal{B}$  basis vectors.

$$L(1, 0, 1) = (2, 1, 1)$$

$$L(2, 1, 1) = (4, 3, 5)$$

$$L(-1, 1, 0) = (-2, 0, 4)$$

Now let's try to find the  $\mathcal{B}$ -coordinates for our results. We can quickly see that

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

but the other two are harder to see. So, let's solve for them.

## The Matrix of a Linear Mapping

### Example

We need to find  $a_1, b_1, c_1 \in \mathbb{R}$  such that

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

We also need to find  $a_2, b_2, c_2 \in \mathbb{R}$  such that

$$a_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$$

Our first equation is equivalent to the system

$$\begin{array}{rcl} a_1 & +2b_1 & -c_1 = 4 \\ & b_1 & +c_1 = 3 \\ a_1 & +b_1 & = 5 \end{array}$$

while our second equation is equivalent to the system

$$\begin{array}{rcl} a_2 & +2b_2 & -c_2 = -2 \\ & b_2 & +c_2 = 0 \\ a_2 & +b_2 & = 4 \end{array}$$

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#### Example

Since these systems have the same coefficient matrix, we can solve them simultaneously by row reducing the following doubly augmented matrix:

$$\begin{aligned} & \left[ \begin{array}{ccc|cc} 1 & 2 & -1 & 4 & -2 \\ 0 & 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 5 & 4 \end{array} \right] \xrightarrow{R_3 - R_1} \left[ \begin{array}{ccc|cc} 1 & 2 & -1 & 4 & -2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & -1 & 1 & 1 & 6 \end{array} \right] \xrightarrow{R_3 + R_2} \left[ \begin{array}{ccc|cc} 1 & 2 & -1 & 4 & -2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 2 & 4 & 6 \end{array} \right] \xrightarrow{\frac{1}{2}R_3} \\ & \sim \left[ \begin{array}{ccc|cc} 1 & 2 & -1 & 4 & -2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{array} \right] \xrightarrow{\begin{array}{l} R_1 + R_3 \\ R_2 - R_3 \end{array}} \left[ \begin{array}{ccc|cc} 1 & 2 & 0 & 6 & 1 \\ 0 & 1 & 0 & 1 & -3 \\ 0 & 0 & 1 & 2 & 3 \end{array} \right] \xrightarrow{R_1 - 2R_2} \left[ \begin{array}{ccc|cc} 1 & 0 & 0 & 4 & 7 \\ 0 & 1 & 0 & 1 & -3 \\ 0 & 0 & 1 & 2 & 3 \end{array} \right] \end{aligned}$$

And so we see that  $a_1 = 4$ ,  $b_1 = 1$ ,  $c_1 = 2$ ,  $a_2 = 7$ ,  $b_2 = -3$ , and  $c_2 = 3$ . This means that

$$\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}_B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}_B = \begin{bmatrix} 7 \\ -3 \\ 3 \end{bmatrix}.$$

And all of this means that  $[L]_B = \begin{bmatrix} 0 & 4 & 7 \\ 1 & 1 & -3 \\ 0 & 2 & 3 \end{bmatrix}$

We've spent a lot of time looking at how to find  ${}_c[L]_B$ , but what do we do with it when we have it? We use it to compute  $[L(\mathbf{x})]_C$  from  $[\mathbf{x}]_B$  of course!

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Let  $L : M(2, 2) \rightarrow P_2$  be defined by  $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b) + (a+c)x + (a+d)x^2$ , and let

$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$  and  $C = \{1, 1+x, 1+x+x^2\}$ , as in our earlier example.

$${}_c[L]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Now, if we have  $[\mathbf{x}]_B = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix}$ , then we get that

$$\begin{aligned} [L(\mathbf{x})]_C &= {}_c[L]_B[\mathbf{x}]_B \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -1 \\ 10 \end{bmatrix} \end{aligned}$$

Then

$$L(\mathbf{x}) = 2(1) - 1(1+x) + 10(1+x+x^2) = 11 + 9x + 10x^2.$$

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#### Example

We can also compute  $L(\mathbf{x})$  by first finding  $\mathbf{x}$ .

Since

$$[\mathbf{x}]_B = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix},$$

we have

$$\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}$$

And we can then compute that

$$L(\mathbf{x}) = L\left(\begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}\right) = (6+5) + (6+3)x + (6+4)x^2 = 11 + 9x + 10x^2$$

This is the same as before.