The Standard Inner Product for $\mathbb{C}^n$

When we first studied $\mathbb{R}^n$, we introduced a dot product for vectors. We will extend this definition to the complex numbers as well.

Definition: Let $\vec{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ be vectors in $\mathbb{C}^n$. Then the dot product of $\vec{z}$ and $\vec{w}$ is

$$\vec{z} \cdot \vec{w} = z_1 w_1 + \cdots + z_n w_n$$

Example

$\begin{bmatrix} 2-i \\ 1+i \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -i \\ 5-i \\ 3+i \end{bmatrix}$
$= (2-i)(-i) + (1+i)(5-i) + (3)(3+i)$
$= -2i + i^2 + 5 - i + 5i - i^2 + 9 + 3i$
$= 14 + 5i$

In $\mathbb{R}^n$, we ended up doing quite a lot with the dot product, but unfortunately much of this does not translate well into the complex numbers.

The exception being the definition of matrix multiplication, where we still have that the jk-th entry of AB is the dot product of the j-th row of A with the k-th column of B, even when A and B have complex entries.

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But if we tried to use the dot product to define length, we would find that

$$\vec{z} \cdot \vec{z} = z_1^2 + \cdots + z_n^2$$

At just a glance, this may seem like a fine start to defining length, but consider that $\vec{z} \cdot \vec{z}$ is a complex number.

So, what would $\sqrt{\vec{z} \cdot \vec{z}}$ be? We get two square roots, as in the real numbers, but in the real numbers we know that one of these is positive and one is negative, and we take the symbol $\sqrt{}$ to mean the positive square root.

But the complex numbers do not have a notion of positive, negative, or even greater than or less than.

So, trying to decide which complex number should be the length of a vector is an impossible task.

Instead, we will want our notion of length to be a real number, and as such the dot product can not be used for this purpose.

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Instead, we will seek inspiration from $\mathbb{C} = \mathbb{C}^1$.

For here, we already have a notion of length, which is the modulus of a complex number.
Recalling that $|z|^2 = z\overline{z}$ gives us the inspiration to define the following:

Definition: In $\mathbb{C}^n$ the standard inner product $\langle\ ,\ \rangle$ is defined by

$$\langle \vec{z}, \vec{w} \rangle = \vec{z} \cdot \overline{\vec{w}} = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n, \text{ for } \vec{w}, \vec{z} \in \mathbb{C}^n$$

Note that if $\vec{z}$ and $\vec{w}$ contained only real entries, then $\overline{w}_j = w_j$, and this inner product is the same as the dot product.

So while it may not have seemed so at first, this is the correct way to extend the definition of the dot product to the complex numbers.
And we can use this to define the length of complex vectors.

Definition: Let $\vec{z}$ be a vector in $\mathbb{C}^n$. Then we define the length of $\vec{z}$ by $\sqrt{\langle \vec{z}, \vec{z} \rangle} = \sqrt{\vec{z} \cdot \overline{\vec{z}}}$

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Example

$$\left\langle \begin{bmatrix} 1+3i \\ -1+2i \end{bmatrix}, \begin{bmatrix} 3+4i \\ 2-i \end{bmatrix} \right\rangle = (1+3i)(3-4i) + (-1+2i)(2+i)$$
$$= 3 - 4i + 9i - 12i^2 - 2 - i + 4i + 2i^2$$
$$= 11 + 8i$$

To calculate $\left\| \begin{bmatrix} 1+3i \\ -1+2i \end{bmatrix} \right\|$, we first calculate $\left\| \begin{bmatrix} 1+3i \\ -1+2i \end{bmatrix} \right\|^2 = \left\langle \begin{bmatrix} 1+3i \\ -1+2i \end{bmatrix}, \begin{bmatrix} 1+3i \\ -1+2i \end{bmatrix} \right\rangle$:

$$\left\langle \begin{bmatrix} 1+3i \\ -1+2i \end{bmatrix}, \begin{bmatrix} 1+3i \\ -1+2i \end{bmatrix} \right\rangle = (1+3i)(1-3i) + (-1+2i)(-1-2i)$$
$$= 1 - 3i + 3i - 9i^2 + 1 + 2i - 2i - 4i^2$$
$$= 15$$

So we have $\left\| \begin{bmatrix} 1+3i \\ -1+2i \end{bmatrix} \right\|^2 = 15$, which means that $\left\| \begin{bmatrix} 1+3i \\ -1+2i \end{bmatrix} \right\| = \sqrt{15}$.

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If we take a closer look at our example, we see that the i terms all canceled out when we took the inner product. This was the point of this definition, but it means that we can simplify our calculation of the length.

For, if $\vec{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 i \\ \vdots \\ x_n + y_n i \end{bmatrix}$ (for $x_j, y_j \in \mathbb{R}$), then we see that

$$\langle \vec{z}, \vec{z} \rangle = \sum_{j=1}^{n} (x_j + y_j i)(x_j - y_j i) = \sum_{j=1}^{n} x_j^2 + y_j^2$$

So, to find the length of $\vec{z}$, we sum the squares of the real parts and the squares of the imaginary parts, and then take the square root.

Example

$$\left\| \begin{bmatrix} 3 + 4i \\ 2 - i \end{bmatrix} \right\| = \sqrt{3^2 + 4^2 + 2^2 + (-1)^2} = \sqrt{30}$$