Bases for Fundamental Subspaces

Last Lecture

- The columns of a matrix A which correspond to the columns in the RREF of A containing leading ones form a basis for $\mathrm{Col}(A)$.

In This Lecture

- We continue to look at finding bases for the fundamental subspaces.

Bases for Fundamental Subspaces

Theorem 7.1.3

Let A be an $m \times n$ matrix. The set of all non-zero rows in the reduced row echelon form of A form a basis for $\mathrm{Row}(A)$. Hence,

$$\dim(\mathrm{Row}(A)) = \mathrm{rank}(A)$$

Proof

Let R be the RREF of A.

By definition of the RREF, we have that the set of all non-zero rows of R form a basis for the row space of R.

Let E be the matrix obtained by multiplying the elementary matrices which row reduce A to R.

Since E is invertible we get that $A = E^{-1}R$.

Then,

$$\begin{aligned}\mathrm{Row}(A) &= \{A^T\vec{x} \mid \vec{x} \in \mathbb{R}^m\} \\ &= \{(E^{-1}R)^T\vec{x} \mid \vec{x} \in \mathbb{R}^m\} \\ &= \{R^T(E^{-1})^T\vec{x} \mid \vec{x} \in \mathbb{R}^m\} \\ &= \{R^T\vec{y} \mid \vec{y} \in \mathbb{R}^m\} \\ &= \mathrm{Row}(R)\end{aligned}$$

Hence, the set of all non-zero rows of R also form a basis for the row space of A. ☐

Corollary 7.1.4

For any $m \times n$ matrix A we have $\mathrm{rank}\,A = \mathrm{rank}\,A^T$.

Bases for Fundamental Subspaces

Example

Find a basis for the rowspace and columnspace of $A = \begin{bmatrix} 2 & -1 & -3 & 0 \\ -2 & 1 & 1 & -4 \\ 4 & -2 & -4 & 4 \end{bmatrix}$.

Solution

Row reducing A gives

$$\begin{bmatrix} 2 & -1 & -3 & 0 \\ -2 & 1 & 1 & -4 \\ 4 & -2 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

A basis for $\mathrm{Col}(A)$ is $\left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix} \right\}$.

A basis for $\mathrm{Row}(A)$ is $\left\{ \begin{bmatrix} 1 \\ -1/2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.

Bases for Fundamental Subspaces

Example

Find a basis for the nullspace of $A = \begin{bmatrix} 2 & -1 & -3 & 0 \\ -2 & 1 & 1 & -4 \\ 4 & -2 & -4 & 4 \end{bmatrix}$.

Solution

To solve the homogeneous system $A\vec{x} = \vec{0}$, we row reduce the coefficient matrix A.

$$\begin{bmatrix} 2 & -1 & -3 & 0 \\ -2 & 1 & 1 & -4 \\ 4 & -2 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rewriting the RREF back as a homogeneous system gives

$$\begin{aligned} x_1 - \tfrac{1}{2}x_2 + 3x_4 &= 0 \\ x_3 + 2x_4 &= 0 \end{aligned}$$

Hence, we get

$$\begin{aligned} x_1 &= \tfrac{1}{2}x_2 - 3x_4 \\ x_3 &= -2x_4 \end{aligned}$$

Thus, every $\vec{x} \in \mathrm{Null}(A)$ satisfies

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \quad x_2, x_4 \in \mathbb{R}$$

Bases for Fundamental Subspaces

Example

Find a basis for the nullspace of $A = \begin{bmatrix} 2 & -1 & -3 & 0 \\ -2 & 1 & 1 & -4 \\ 4 & -2 & -4 & 4 \end{bmatrix}$.

Solution

$$\begin{bmatrix} 2 & -1 & -3 & 0 \\ -2 & 1 & 1 & -4 \\ 4 & -2 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, by Theorem 2.2.5, $\mathcal{B} = \left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for $\operatorname{Null}(A)$.

Note the pattern between the RREF of A and the basis of the nullspace of A.

Bases for Fundamental Subspaces

Example

Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 5 \\ 1 & 3 & 5 \\ 1 & 1 & 3 \end{bmatrix}$. Find a basis for the four fundamental subspaces.

Solution

Row reducing A we get $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 5 \\ 1 & 3 & 5 \\ 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R$

Thus, a basis for $\operatorname{Row}(A)$ is the set of non-zero rows of R. That is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

A basis for $\operatorname{Col}(A)$ is the columns of A which correspond to the columns of R which contain leading ones. Hence, a basis for $\operatorname{Col}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$.

From the reduced row echelon form, we can see that a basis for $\operatorname{Null}(A)$ is $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Bases for Fundamental Subspaces

Example

Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 5 \\ 1 & 3 & 5 \\ 1 & 1 & 3 \end{bmatrix}$. Find a basis for the four fundamental subspaces.

Solution

To find a basis for $\mathrm{Null}(A^T)$, we can just use our method for finding the nullspace of a matrix on A^T. That is, we row reduce A^T to get

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 3 & 5 & 5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for the left nullspace of A is $\left\{ \begin{bmatrix} -5 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Bases for Fundamental Subspaces

Example

You may have noticed the following:

- Combining the basis for $\mathrm{Col}(A)$ with the basis for $\mathrm{Null}(A^T)$ gives a basis for $\mathbb{R}^4$.
- Combining the basis for $\mathrm{Row}(A)$ with the basis for $\mathrm{Null}(A)$ gives a basis for $\mathbb{R}^3$.

Also note that it is not necessary to row-reduce A^T to find a basis for $\mathrm{Null}(A^T)$.
We can use the fact that $EA = R$, where E is the product of elementary matrices used to bring A to R, to find a basis for $\mathrm{Null}(A^T)$. This is left as an exercise.

Dimension Theorem

Theorem 7.1.5 - Dimension Theorem

If A is an $m \times n$ matrix, then

$$\operatorname{rank}(A) + \dim\big(\operatorname{Null}(A)\big) = n$$

Sketch of Proof

To prove the Dimension Theorem, we only need to show that

$$\dim\big(\operatorname{Null}(A)\big) = n - \operatorname{rank}(A)$$

But this approach turns out to be notationally ugly.

Instead, we will prove that

$$\operatorname{rank}(A) = n - \dim\big(\operatorname{Null}(A)\big)$$

Dimension Theorem

Theorem 7.1.5 - Dimension Theorem

If A is an $m \times n$ matrix, then

$$\operatorname{rank}(A) + \dim\big(\operatorname{Null}(A)\big) = n$$

Proof

Assume that $\dim\big(\operatorname{Null}(A)\big) = k$. Then there exists a basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ for $\operatorname{Null}(A)$.
We can extend $\mathcal{B}$ to a basis $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$.
By definition of $\operatorname{Col}(A)$, we have that $\mathcal{C} = \{A\vec{v}_{k+1}, \dots, A\vec{v}_n\}$ is a set of $n-k$ vectors in $\operatorname{Col}(A)$.
Consider

$$\begin{aligned}\vec{0} &= c_{k+1}A\vec{v}_{k+1} + \cdots + c_nA\vec{v}_n \\ &= A(c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n)\end{aligned}$$

Then, $c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n \in \operatorname{Null}(A)$.
Hence, we can write this vector as a linear combination of the vectors in $\mathcal{B}$ since $\mathcal{B}$ is a basis for $\operatorname{Null}(A)$.
Say,

$$\begin{aligned}c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n &= d_1\vec{v}_1 + \cdots + d_k\vec{v}_k \\ -d_1\vec{v}_1 - \cdots - d_k\vec{v}_k + c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n &= \vec{0}\end{aligned}$$

Since $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent we get $d_1 = \cdots = d_k = c_{k+1} = \cdots = c_n = 0$.
So, $\mathcal{C}$ is linearly independent.

Dimension Theorem

Theorem 7.1.5 - Dimension Theorem

If A is an $m \times n$ matrix, then

$$\operatorname{rank}(A) + \dim\big(\operatorname{Null}(A)\big) = n$$

Proof

Let $\vec{b} \in \operatorname{Col}(A)$.

Then there exists $\vec{x} \in \mathbb{R}^n$ such that

$$\begin{aligned}
\vec{b} &= A\vec{x} \\
&= A(x_1\vec{v}_1 + \cdots + x_k\vec{v}_k + x_{k+1}\vec{v}_{k+1} + \cdots + x_n\vec{v}_n) \\
&= x_1A\vec{v}_1 + \cdots + x_kA\vec{v}_k + x_{k+1}A\vec{v}_{k+1} + \cdots + x_nA\vec{v}_n \\
&= \vec{0} + \cdots + \vec{0} + x_{k+1}A\vec{v}_{k+1} + \cdots + x_nA\vec{v}_n \\
&= x_{k+1}A\vec{v}_{k+1} + \cdots + x_nA\vec{v}_n
\end{aligned}$$

That is, $\operatorname{Span}\{A\vec{v}_{k+1}, \ldots, A\vec{v}_n\} = \operatorname{Col}(A)$.

Therefore, we have shown that $\{A\vec{v}_{k+1}, \ldots, A\vec{v}_n\}$ is a basis for $\operatorname{Col}(A)$ and hence

$$\dim\big(\operatorname{Col}(A)\big) = n - k = n - \dim\big(\operatorname{Null}(A)\big)$$

as required.

□