Complex Number Review

Definition: We define i to be a number such that $i^2 = -1$ and define the set of complex numbers to be

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

For a complex number $z = a + bi$, we define the real part of z by

$$\text{Re}(z) = a$$

and the imaginary part of z by

$$\text{Im}(z) = b$$

If $b = 0$, then we say that z is real. If $a = 0$ and $b \neq 0$, then we say that z is imaginary.

Now we define common operations on complex numbers:

- We define addition of two complex numbers $a + bi$ and $c + di$ by
$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
- We also define real scalar multiplication of $a + bi$ by $t \in \mathbb{R}$ by
$$t(a + bi) = ta + tbi$$

Thus, the set $\mathbb{C}$ with these operations satisfies our ten familiar propoerties. And so it is a real vector space!

We observe that every vector $z \in \mathbb{C}$ is a linear combination of 1 and i, and that the set $\{1, i\}$ is linearly independent.

Thus, the standard basis for $\mathbb{C}$ as a real vector space is $\{1, i\}$. Hence, it is a 2-dimensional real vector space.

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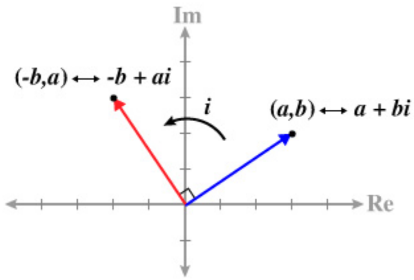
We know that all 2-dimensionsal real vector spaces are isomporhic.

In particular, we have that $\mathbb{C}$ is isomporphic to the plane, and so we often talk about the *complex plane*.

Normally, we relate the complex number $z = a + bi \in \mathbb{C}$ to the point $(a, b) \in \mathbb{R}^2$.



Can we multiply complex numbers by other complex numbers? Consider a simple example:

We want

$$i(a + bi) = -b + ai$$

This looks like a linear mapping named i that takes the vector $a + bi$ and outputs the vector $-b + ai$.

We can find the matrix of this linear mapping with respect to the basis $S = \{1, i\}$. We get that

$$[i]_S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We recognize this as a rotation by $\frac{\pi}{2}$, as seen in the picture above.

Similarly, we can show that multiplying by any $\alpha \in \mathbb{C}$ corresponds to a rotation and a stretch.

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We define multiplication of two complex numbers $a+bi$ and $c+di$ by

$$(a+bi)(c+di) = ac + adi + bci + bdi^2 = ac - bd + (ad+bc)i$$

Example

Evaluate the folowing:

(a) $2-3i+3+i = (2+3)+(-3+1)i = 5-2i$

(b) $(2-i)(3+2i) = [2(3)-(-1)(2)]+[(2(2)+(-1)(3)]i = 8+i$

(c) $(2-i)(2+i) = [2(2)-(-1)(1)]+[2(1)+(-1)(2)]i = 5$

We have the following familiar properties for basic operations on complex numbers:

Theorem 11.1.1

If $z_1, z_2, z_3 \in \mathbb{C}$, then

1. $z_1 + z_2 = z_2 + z_1$ — addition is commutative
2. $z_1 z_2 = z_2 z_1$ — multiplication is commutative
3. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ — addition is associative
4. $z_1(z_2 z_3) = (z_1 z_2) z_3$ — multiplication is associative
5. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ — multiplication is distributive over addition

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Definition: Let $z = a+bi \in \mathbb{C}$. The complex conjugate of z is $\overline{z} = a - bi$.

Example

$$\overline{3-4i} = 3+4i \qquad \overline{2i} = -2i \qquad \overline{-5} = -5$$

Theorem 11.1.2

If $z = a+bi, z_1, z_2 \in \mathbb{C}$, then

1. $\overline{\overline{z}} = z$
2. z is real if and only if $\overline{z} = z$
3. If $z \neq 0$, then z is imaginary if and only if $\overline{z} = -z$
4. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
5. $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
6. $z + \overline{z} = 2\,\mathrm{Re}(z) = 2a$
7. $z - \overline{z} = 2i\,\mathrm{Im}(z) = 2bi$
8. $z\overline{z} = a^2 + b^2$

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We observe that $a^2 + b^2$ is the square of the Euclidean length of the point (a, b) from the origin in $\mathbb{R}^2$. This motivates the following definition:

Definition: Let $z = a + bi \in \mathbb{C}$. We define the absolute value of z to be $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$.

Theorem 11.1.3

If $w, z \in \mathbb{C}$, then

1. $|z| \in \mathbb{R}$ and $|z| \geq 0$
2. $|z| = 0$ if and only if $z = 0$
3. $|wz| = |w|\,|z|$
4. $|w + z| \leq |w| + |z|$

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Finally, can define division of two complex numbers:
For any $w, z \in \mathbb{C}$ with $z \neq 0$ we have

$$\frac{w}{z} = \frac{w\bar{z}}{z\bar{z}} = \frac{w\bar{z}}{|z|^2}$$

Example

Calculate $\dfrac{4 + 3i}{2 - 3i}$.

Solution

$$\frac{4 + 3i}{2 - 3i} = \frac{(4 + 3i)(2 + 3i)}{(2 - 3i)(2 + 3i)} = \frac{-1 + 18i}{13} = -\frac{1}{13} + \frac{18}{13}i$$

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We can also row reduce matrices with complex entries! That is, we can solve systems of linear equations with complex entries.

Row reduction is the same as in the real case, except that the calculations are a little more… *complex*!

Also, free variables can take on any **complex** value.

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Example

Find the solution set for the following homogeneous system of linear equations:

$$\begin{aligned} z_1 + 2iz_2 + (1+3i)z_3 &= 0 \\ iz_1 + (-2+i)z_2 + (-3+3i)z_3 &= 0 \\ -2z_1 + 3iz_2 + (-2+8i)z_3 &= 0 \end{aligned}$$

Solution

We row reduce the coefficient matrix of the system to RREF:

$$\begin{bmatrix} 1 & 2i & 1+3i \\ i & -2+i & -3+3i \\ -2 & 3i & -2+8i \end{bmatrix} \begin{matrix} \\ R_2 - iR_1 \\ R_3 + 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2i & 1+3i \\ 0 & i & 2i \\ 0 & 7i & 14i \end{bmatrix} (-i)R_2$$

$$\sim \begin{bmatrix} 1 & 2i & 1+3i \\ 0 & 1 & 2 \\ 0 & 7i & 14i \end{bmatrix} \begin{matrix} R_1 - 2iR_2 \\ \\ R_3 - 7iR_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1-i \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, we see that z_3 is a free variable. Hence, we let $z_3 = \alpha \in \mathbb{C}$.

Then, a vector equation for the solution set is

$$\vec{z} = \alpha \begin{bmatrix} -1+i \\ -2 \\ 1 \end{bmatrix}, \quad \alpha \in \mathbb{C}$$

We can verify that if we take $\alpha = 1$, then $z_1 = -1+i$, $z_2 = -2$, and $z_3 = 1$, does satisfy the original system.