Fundamental Subspaces of a Matrix

In This Lecture

- We will review the definition of the four fundamental subspaces of a matrix.
- Prove a theorem which gives us an easy way to find a basis for the columnspace of a matrix.

Fundamental Subspaces of a Matrix

Definition: Let A be an $m \times n$ matrix. The four fundamental subspaces of A are as follows:

1. The columnspace of A is defined by $\mathrm{Col}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$.
2. The rowspace of A is defined by $\mathrm{Row}(A) = \{A^T\vec{x} \mid \vec{x} \in \mathbb{R}^m\} = \mathrm{Col}(A^T)$.
3. The nullspace of A is defined by $\mathrm{Null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$.
4. The left nullspace of A is defined by $\mathrm{Null}(A^T) = \{\vec{x} \in \mathbb{R}^m \mid A^T\vec{x} = \vec{0}\}$.

Theorem 7.1.1

If A is an $m \times n$ matrix, then $\mathrm{Col}(A)$ and $\mathrm{Null}(A^T)$ are subspaces of $\mathbb{R}^m$, and $\mathrm{Row}(A)$ and $\mathrm{Null}(A)$ are subspaces of $\mathbb{R}^n$.

Theorem 7.1.2

Let A be an $m \times n$ matrix. The columns of A which correspond to leading ones in the reduced row echelon form of A form a basis for $\text{Col}(A)$. Moreover,

$$\dim \text{Col}(A) = \text{rank}(A)$$

Proof

If A is the zero matrix, then the result is trivial.
Assume that $\text{rank}(A) = r > 0$.

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Proof

If A is the zero matrix, then the result is trivial.
Assume that $\text{rank}(A) = r > 0$.

Denote the columns of the reduced row echelon form (RREF) R of A by $\vec{r}_1, \dots, \vec{r}_n$.
Since $\text{rank}(A) = r$, R contains r leading ones.
Let $t_1, \dots, t_r$ denote the indices of the columns of R which contain leading ones.

We will show $\mathcal{B} = \{\vec{r}_{t_1}, \dots, \vec{r}_{t_r}\}$ is a basis for $\text{Col}(R)$.

Example

$$A = \begin{bmatrix} 1 & 1 & 3 & -1 \\ 2 & 2 & 1 & 3 \\ 1 & 1 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$$\vec{r}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{r}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{r}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{r}_4 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

We have that $t_1 = 1$ and $t_2 = 3$ as these are the indices of the columns of R containing leading 1s.

$\mathcal{B} = \{\vec{r}_{t_1}, \vec{r}_{t_2}\} = \{\vec{r}_1, \vec{r}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\text{Col}(R)$.

Theorem 7.1.2

Let A be an $m \times n$ matrix. The columns of A which correspond to leading ones in the reduced row echelon form of A form a basis for $\operatorname{Col}(A)$. Moreover,

$$\dim \operatorname{Col}(A) = \operatorname{rank}(A)$$

Proof

By definition of RREF, the vectors $\vec{r}_{t_1}, \dots, \vec{r}_{t_r}$ are standard basis vectors of $\mathbb{R}^n$ and hence form a linearly independent set.

Moreover, every column of R which does not contain a leading one can be written as a linear combination of the columns which do contain leading ones.

So, we also have $\operatorname{Span} \mathcal{B} = \operatorname{Col}(R)$.
Therefore, $\mathcal{B}$ is a basis for $\operatorname{Col}(R)$ as claimed.

Example

$$A = \begin{bmatrix} 1 & 1 & 3 & -1 \\ 2 & 2 & 1 & 3 \\ 1 & 1 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$$\vec{r}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{r}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{r}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{r}_4 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

We have that $t_1 = 1$ and $t_2 = 3$ as these are the indices of the columns of R containing leading 1s.

$\mathcal{B} = \{\vec{r}_{t_1}, \vec{r}_{t_2}\} = \{\vec{r}_1, \vec{r}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\operatorname{Col}(R)$.

Theorem 7.1.2

Let A be an $m \times n$ matrix. The columns of A which correspond to leading ones in the reduced row echelon form of A form a basis for $\operatorname{Col}(A)$. Moreover,

$$\dim \operatorname{Col}(A) = \operatorname{rank}(A)$$

Proof

By definition of RREF, the vectors $\vec{r}_{t_1}, \dots, \vec{r}_{t_r}$ are standard basis vectors of $\mathbb{R}^n$ and hence form a linearly independent set.

Moreover, every column of R which does not contain a leading one can be written as a linear combination of the columns which do contain leading ones.

So, we also have $\operatorname{Span} \mathcal{B} = \operatorname{Col}(R)$.
Therefore, $\mathcal{B}$ is a basis for $\operatorname{Col}(R)$ as claimed.

Denote the columns of A by $\vec{a}_1, \dots, \vec{a}_n$.

We will show $\mathcal{C} = \{\vec{a}_{t_1}, \dots, \vec{a}_{t_r}\}$ is a basis for $\operatorname{Col}(A)$ using the fact that $\mathcal{B}$ is a basis for $\operatorname{Col}(R)$.

Example

$$A = \begin{bmatrix} 1 & 1 & 3 & -1 \\ 2 & 2 & 1 & 3 \\ 1 & 1 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$$\vec{r}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{r}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{r}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{r}_4 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

We have that $t_1 = 1$ and $t_2 = 3$ as these are the indices of the columns of R containing leading 1s.

$\mathcal{B} = \{\vec{r}_{t_1}, \vec{r}_{t_2}\} = \{\vec{r}_1, \vec{r}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\operatorname{Col}(R)$.

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \vec{a}_4 = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$$

Theorem 7.1.2

Let A be an $m \times n$ matrix. The columns of A which correspond to leading ones in the reduced row echelon form of A form a basis for $\operatorname{Col}(A)$. Moreover,

$$\dim \operatorname{Col}(A) = \operatorname{rank}(A)$$

Proof

By definition of RREF, the vectors $\vec{r}_{t_1}, \dots, \vec{r}_{t_r}$ are standard basis vectors of $\mathbb{R}^n$ and hence form a linearly independent set.

Moreover, every column of R which does not contain a leading one can be written as a linear combination of the columns which do contain leading ones.

So, we also have $\operatorname{Span} \mathcal{B} = \operatorname{Col}(R)$.
Therefore, $\mathcal{B}$ is a basis for $\operatorname{Col}(R)$ as claimed.

Denote the columns of A by $\vec{a}_1, \dots, \vec{a}_n$.

We will show $\mathcal{C} = \{\vec{a}_{t_1}, \dots, \vec{a}_{t_r}\}$ is a basis for $\operatorname{Col}(A)$ using the fact that $\mathcal{B}$ is a basis for $\operatorname{Col}(R)$.

Example

$$A = \begin{bmatrix} 1 & 1 & 3 & -1 \\ 2 & 2 & 1 & 3 \\ 1 & 1 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$$\vec{r}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{r}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{r}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{r}_4 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

We have that $t_1 = 1$ and $t_2 = 3$ as these are the indices of the columns of R containing leading 1s.

$\mathcal{B} = \{\vec{r}_{t_1}, \vec{r}_{t_2}\} = \{\vec{r}_1, \vec{r}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\operatorname{Col}(R)$.

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \vec{a}_4 = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$$

$\vec{a}_{t_1} = \vec{a}_1$, $\vec{a}_{t_2} = \vec{a}_3$.

Show $\mathcal{C} = \{\vec{a}_{t_1}, \vec{a}_{t_2}\} = \{\vec{a}_1, \vec{a}_3\}$ is a basis for $\operatorname{Col}(A)$.

Theorem 7.1.2

Let A be an $m \times n$ matrix. The columns of A which correspond to leading ones in the reduced row echelon form of A form a basis for $\operatorname{Col}(A)$. Moreover,

$$\dim \operatorname{Col}(A) = \operatorname{rank}(A)$$

Proof

By definition of RREF, the vectors $\vec{r}_{t_1}, \dots, \vec{r}_{t_r}$ are standard basis vectors of $\mathbb{R}^n$ and hence form a linearly independent set.

Moreover, every column of R which does not contain a leading one can be written as a linear combination of the columns which do contain leading ones.

So, we also have $\operatorname{Span} \mathcal{B} = \operatorname{Col}(R)$.
Therefore, $\mathcal{B}$ is a basis for $\operatorname{Col}(R)$ as claimed.

Denote the columns of A by $\vec{a}_1, \dots, \vec{a}_n$.

We will show $\mathcal{C} = \{\vec{a}_{t_1}, \dots, \vec{a}_{t_r}\}$ is a basis for $\operatorname{Col}(A)$ using the fact that $\mathcal{B}$ is a basis for $\operatorname{Col}(R)$.

To do this, we need to find a relationship between the vectors in $\mathcal{B}$ and $\mathcal{C}$.

Example

$$A = \begin{bmatrix} 1 & 1 & 3 & -1 \\ 2 & 2 & 1 & 3 \\ 1 & 1 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$$\vec{r}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{r}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{r}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{r}_4 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

We have that $t_1 = 1$ and $t_2 = 3$ as these are the indices of the columns of R containing leading 1s.

$\mathcal{B} = \{\vec{r}_{t_1}, \vec{r}_{t_2}\} = \{\vec{r}_1, \vec{r}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\operatorname{Col}(R)$.

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \vec{a}_4 = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$$

$\vec{a}_{t_1} = \vec{a}_1$, $\vec{a}_{t_2} = \vec{a}_3$.

Show $\mathcal{C} = \{\vec{a}_{t_1}, \vec{a}_{t_2}\} = \{\vec{a}_1, \vec{a}_3\}$ is a basis for $\operatorname{Col}(A)$.

Theorem 7.1.2

Let A be an $m \times n$ matrix. The columns of A which correspond to leading ones in the reduced row echelon form of A form a basis for $\mathrm{Col}(A)$. Moreover,

$$\dim \mathrm{Col}(A) = \mathrm{rank}(A)$$

Proof (continued)

Since R is the reduced row echelon form of A there exists a sequence of elementary matrices $E_1, \ldots, E_k$ such that $E_k \cdots E_1 A = R$. Let $E = E_k \cdots E_1$.
Recall that every elementary matrix is invertible, hence $E^{-1} = E_1^{-1} \cdots E_k^{-1}$ exists.
Then

$$R = EA = [E\vec{a}_1 \ \cdots \ E\vec{a}_n]$$

Then $\vec{r}_i = E\vec{a}_i$, or $\vec{a}_i = E^{-1}\vec{r}_i$.

Consider

$$c_1\vec{a}_{t_1} + \cdots + c_r\vec{a}_{t_r} = \vec{0}$$

Multiply both sides by E to get

$$\begin{aligned} E(c_1\vec{a}_{t_1} + \cdots + c_r\vec{a}_{t_r}) &= E\vec{0} \\ c_1E\vec{a}_{t_1} + \cdots + c_rE\vec{a}_{t_r} &= \vec{0} \\ c_1\vec{r}_{t_1} + \cdots + c_r\vec{r}_{t_r} &= \vec{0} \end{aligned}$$

$c_1 = \cdots = c_r = 0$ since $\{\vec{r}_{t_1}, \ldots, \vec{r}_{t_r}\}$ is linearly independent.
Thus, C is linearly independent.

Theorem 7.1.2

Let A be an $m \times n$ matrix. The columns of A which correspond to leading ones in the reduced row echelon form of A form a basis for $\mathrm{Col}(A)$. Moreover,

$$\dim \mathrm{Col}(A) = \mathrm{rank}(A)$$

Proof (continued)

Pick $\vec{b} \in \mathrm{Col}(A)$. Then there exists a $\vec{x} \in \mathbb{R}^n$ such that

$$\begin{aligned} \vec{b} &= A\vec{x} \\ \vec{b} &= E^{-1}R\vec{x} \\ E\vec{b} &= R\vec{x} \end{aligned}$$

Therefore, $E\vec{b}$ is in the columnspace of R and can be written as a linear combination of the basis vectors $\{\vec{r}_{t_1}, \ldots, \vec{r}_{t_r}\}$. Hence, we have $d_1, \ldots, d_r \in \mathbb{R}$ such that

$$\begin{aligned} E\vec{b} &= d_1\vec{r}_{t_1} + \cdots + d_r\vec{r}_{t_r} \\ \vec{b} &= E^{-1}(d_1\vec{r}_{t_1} + \cdots + d_r\vec{r}_{t_r}) \\ \vec{b} &= d_1E^{-1}\vec{r}_{t_1} + \cdots + d_rE^{-1}\vec{r}_{t_r} \\ \vec{b} &= d_1\vec{a}_{t_1} + \cdots + d_r\vec{a}_{t_r} \end{aligned}$$

as required.
Thus, we also have $\mathrm{Span}\{\vec{a}_{t_1}, \ldots, \vec{a}_{t_r}\} = \mathrm{Col}(A)$.
Therefore, $\{\vec{a}_{t_1}, \ldots, \vec{a}_{t_r}\}$ is a basis for $\mathrm{Col}(A)$.
Moreover, this implies that $\dim \mathrm{Col}(A) = r = \mathrm{rank}(A)$. $\square$