Length and Orthogonality

Last Lecture

- We wanted to find nice bases like the standard basis in $\mathbb{R}^n$ (orthogonal and unit length).
- We defined the concept of an inner product on a vector space.

In This Lecture

- We will be able to define length and orthogonality.

Length and Orthogonality

Theorem 9.1.1

If $\mathbb{V}$ is an inner product space with inner product $\langle\, , \rangle$, then for any $\vec{v} \in \mathbb{V}$ we have $\langle \vec{v}, \vec{0} \rangle = 0$.

Definition: Let $\mathbb{V}$ be an inner product space with inner product $\langle\, , \rangle$. The length of $\vec{v} \in \mathbb{V}$ is defined by

$$||\vec{v}|| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$. Find $||A||$ and $||B||$.

Recall that the standard inner product on $M_{2\times 2}(\mathbb{R})$ is defined by $\langle A, B \rangle = \mathrm{tr}(B^T A) = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$.

Solution

By definition, we get

$$||A|| = \sqrt{\langle A, A \rangle} = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$$
$$||B|| = \sqrt{\langle B, B \rangle} = \sqrt{2^2 + (-1)^2 + 0^2 + 3^2} = \sqrt{14}$$

Length and Orthogonality

Example

In $\mathbb{R}^3$ under the standard inner product, we have that the length of $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is 1.

However, under the inner product $\langle \vec{x}, \vec{y} \rangle = ax_1y_1 + x_2y_2 + x_3y_3$ where $a > 0$, then

$$||\vec{e}_1|| = \sqrt{\langle \vec{e}_1, \vec{e}_1 \rangle} = \sqrt{a(1)^2 + 0^2 + 0^2} = \sqrt{a}$$

Example

Find the length of 1 in $P_2(\mathbb{R})$ under the inner product defined by $\langle p(x), q(x) \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$.

Solution

We have the length of $p(x) = 1$ is

$$||1|| = \sqrt{\langle 1, 1 \rangle} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

Length and Orthogonality

Definition: Let $\mathbb{V}$ be an inner product space with inner product $\langle \, , \rangle$. If $\vec{v} \in \mathbb{V}$ with $\|\vec{v}\| = 1$, then $\vec{v}$ is called a unit vector.

Theorem 9.2.1

Let $\vec{v}, \vec{w}$ be any two vectors in an inner product space $\mathbb{V}$ and $t \in \mathbb{R}$. Then

1. $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$
2. $\|t\vec{v}\| = |t|\|\vec{v}\|$
3. $|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\|\|\vec{w}\|$
4. $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$

In many situations, we will be given a vector $\vec{v}$ in an inner product space $\mathbb{V}$ and need to find a unit vector $\hat{v}$ in the direction of $\vec{v}$.
This is called normalizing the vector.
Theorem 9.2.1 part 2 shows us that $\hat{v} = \frac{1}{\|\vec{v}\|}\vec{v}$.

Definition: Let $\mathbb{V}$ be an inner product space. If $\vec{v}, \vec{w} \in \mathbb{V}$ such that $\langle \vec{v}, \vec{w} \rangle = 0$, then we say that $\vec{v}$ and $\vec{w}$ are orthogonal. If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a set in $\mathbb{V}$ such that $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ for all $i \neq j$, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is called an orthogonal set.

Length and Orthogonality

Example

In $P_2(\mathbb{R})$, define the inner product by

$$\langle p(x), q(x)\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$

Determine if $p(x) = x$ and $q(x) = 3x^2 - 2$ are orthogonal.

Solution

We have

$$\langle x, 3x^2 - 2\rangle = (-1)(1) + 0(-2) + (1)(1) = 0$$

Thus, x and $3x^2 - 2$ are orthogonal.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 1 \\ 1 & 7 \end{bmatrix}$. Is the set $\{A, B, C\}$ orthogonal in $M_{2\times 2}(\mathbb{R})$?

Solution

We have

$$\begin{aligned} \langle A, B\rangle &= 1(-1) + 2(2) + 3(-1) + (-1)(0) = 0 \\ \langle A, C\rangle &= 1(2) + 2(1) + 3(1) + (-1)(7) = 0 \\ \langle B, C\rangle &= (-1)(2) + 2(1) + (-1)(1) + 0(7) = -1 \end{aligned}$$

So, B and C are not orthogonal. Therefore, $\{A, B, C\}$ is not orthogonal in $M_{2\times 2}(\mathbb{R})$.

Length and Orthogonality

Example

Show that in $\mathbb{R}^2$ under the inner product defined by

$$\langle \vec{x}, \vec{y}\rangle = 2x_1y_1 - 2x_1y_2 - 2x_2y_1 + 4x_2y_2$$

the vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are not orthogonal.

Solution

We have

$$\langle \vec{e}_1, \vec{e}_2\rangle = 2(1)(0) - 2(1)(1) - 2(0)(0) + 4(0)(1) = -2$$

So, they are not orthogonal.

Length and Orthogonality

Theorem 9.2.2

Let $\mathbb{V}$ be an inner product space. If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is an orthogonal set in $\mathbb{V}$, then

$$||\vec{v}_1 + \cdots + \vec{v}_k||^2 = ||\vec{v}_1||^2 + \cdots + ||\vec{v}_k||^2$$

The proof is left as an exercise.

Length and Orthogonality

Theorem 9.2.3

Let $\mathbb{V}$ be an inner product space. If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is an orthogonal set of non-zero vectors in $\mathbb{V}$, then the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

Proof

Consider

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$$

Then, we get

$$\begin{aligned} 0 &= \langle 0, \vec{v}_i \rangle = \langle c_1\vec{v}_1 + \cdots + c_k\vec{v}_k, \vec{v}_i \rangle \\ &= c_1\langle \vec{v}_1, \vec{v}_i \rangle + \cdots + c_{i-1}\langle \vec{v}_{i-1}, \vec{v}_i \rangle + c_i\langle \vec{v}_i, \vec{v}_i \rangle + c_{i+1}\langle \vec{v}_{i+1}, \vec{v}_i \rangle + \cdots + c_k\langle \vec{v}_k, \vec{v}_i \rangle \\ &= 0 + \cdots + 0 + c_i||\vec{v}_i||^2 + 0 + \cdots + 0 \end{aligned}$$

But, $\vec{v}_i \neq \vec{0}$, so $||\vec{v}_i|| \neq 0$ and thus $c_i = 0$. Since this is valid for $1 \leq i \leq k$, we get that $c_1 = \cdots = c_k = 0$ is the only solution, so $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

$\square$

This theorem shows us that if we have an orthogonal set of n non-zero vectors in an n-dimensional inner product space $\mathbb{V}$, then the set is a basis for $\mathbb{V}$.

Length and Orthogonality

Definition: Let $\mathbb{V}$ be an inner product space. If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthogonal set in $\mathbb{V}$ that is a basis for $\mathbb{V}$, then we call $\{\vec{v}_1, \dots, \vec{v}_n\}$ an orthogonal basis for $\mathbb{V}$.

Definition: Let $\mathbb{V}$ be an inner product space. If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthogonal set of unit vectors in $\mathbb{V}$ that is a basis for $\mathbb{V}$, then we call $\{\vec{v}_1, \dots, \vec{v}_n\}$ an orthonormal basis for $\mathbb{V}$.