Normal Matrices

In the last lecture we proved that Hermitian, skew-Hermitian, and unitary matrices are all unitarily diagonalizable.
However, these are only a very small portion of the matrices that are unitarily diagonalizable.
We want to derive the condition which is equivalent to unitary diagonalizability.
How should we go about this?

Instead of the guess and check approach we tried in the last lecture, we really should have been doing the same thing that we did in the real case: work backwards.
We should assume that A is unitarily diagonalizable and see what condition that puts on A.

Normal Matrices

Assume that A is unitarily diagonalizable. Then, there exists a unitary matrix U such that

$$U^*AU = D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

Recall that the strategy for the three proofs in the last lecture was to show that because the upper triangular matrix T was similar to A, it had the same property as A.
Thinking about the proofs in the last lecture we observe that

$$DD^* = \operatorname{diag}(\lambda_1, \dots, \lambda_n)\operatorname{diag}(\overline{\lambda_1}, \dots, \overline{\lambda_n}) = \operatorname{diag}(|\lambda_1|^2, \dots, |\lambda_n|^2) = D^*D$$

Since $U^*AU = D$, we have that $A = UDU^*$ and so

$$\begin{aligned} AA^* &= (UDU^*)(UDU^*)^* \\ &= UDU^*UD^*U^* \\ &= UDD^*U^* \\ &= UD^*DU^* \\ &= UD^*U^*UDU^* \\ &= (UDU^*)^*(UDU^*) \\ &= A^*A \end{aligned}$$

Definition: An $n \times n$ matrix A such that $AA^* = A^*A$ is called normal.

Normal Matrices

Theorem 11.5.9 - The Spectral Theorem for Normal Matrices

A matrix $A \in M_{n\times n}(\mathbb{C})$ is unitarily diagonalizable if and only if it is normal.

Proof

We already proven that every unitarily diagonalizable matrix is normal.
To prove the other direction we, of course, use the same strategy that we were using for the proofs in the last lecture.

By Schur's theorem, there exists a unitary matrix U such that $U^*AU = T$ is upper triangular.
Observe that

$$TT^* = (U^*AU)(U^*A^*U) = U^*AA^*U = U^*A^*AU = (U^*A^*U)(U^*AU) = T^*T$$

Hence, T is also normal.

Normal Matrices

Theorem 11.5.9 - The Spectral Theorem for Normal Matrices

A matrix $A \in M_{n\times n}(\mathbb{C})$ is unitarily diagonalizable if and only if it is normal.

Proof

Write out T as $T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$.
We next compare the diagonal entries of TT^* and T^*T.

$$\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}\begin{bmatrix} \overline{t_{11}} & 0 & \cdots & 0 \\ \overline{t_{12}} & \overline{t_{22}} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \overline{t_{1n}} & \overline{t_{2n}} & \cdots & \overline{t_{nn}} \end{bmatrix} = \begin{bmatrix} \overline{t_{11}} & 0 & \cdots & 0 \\ \overline{t_{12}} & \overline{t_{22}} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \overline{t_{1n}} & \overline{t_{2n}} & \cdots & \overline{t_{nn}} \end{bmatrix}\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$$

Comparing $1, 1$-entries gives

$$|t_{11}|^2 + |t_{12}|^2 + \cdots + |t_{1n}|^2 = |t_{11}|^2$$

Notice that this is only possible if $t_{12} = \cdots = t_{1n} = 0$.

Therefore, we have $T = \begin{bmatrix} t_{11} & 0 & \cdots & 0 \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$.

Normal Matrices

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Proof

Write out T as $T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$.

We next compare the diagonal entries of TT^* and T^*T.

$$\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}\begin{bmatrix} \overline{t_{11}} & 0 & \cdots & 0 \\ \overline{t_{12}} & \overline{t_{22}} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \overline{t_{1n}} & \overline{t_{2n}} & \cdots & \overline{t_{nn}} \end{bmatrix} = \begin{bmatrix} \overline{t_{11}} & 0 & \cdots & 0 \\ \overline{t_{12}} & \overline{t_{22}} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \overline{t_{1n}} & \overline{t_{2n}} & \cdots & \overline{t_{nn}} \end{bmatrix}\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$$

Next, we compare the $2,2$-entries of TT^* and T^*T.
We get

$$|t_{22}|^2 + \cdots + |t_{2n}|^2 = |t_{22}|^2$$

Hence, $t_{22} = \cdots = t_{2n} = 0$.
Continuing in this way we get that T is diagonal as required.

□

Normal Matrices

Example

Determine which of the following matrices are normal:

(a) $A = \begin{bmatrix} 1 & -i \\ i & i \end{bmatrix}$

Solution

We have

$$\begin{aligned} AA^* &= \begin{bmatrix} 1 & -i \\ i & i \end{bmatrix}\begin{bmatrix} 1 & -i \\ i & -i \end{bmatrix} = \begin{bmatrix} 2 & -1-i \\ -1-i & 2 \end{bmatrix} \\ A^*A &= \begin{bmatrix} 1 & -i \\ i & -i \end{bmatrix}\begin{bmatrix} 1 & -i \\ i & i \end{bmatrix} = \begin{bmatrix} 2 & 1-i \end{bmatrix} \end{aligned}$$

At this point, we can already see that $AA^* \neq A^*A$, so A is not normal.

(b) $B = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$

Solution

We observe that B is Hermitian.
Thus, B is unitarily diagonalizable and so it is normal.

Normal Matrices

Example

Determine which of the following matrices are normal:

(c) $C = \begin{bmatrix} 1+i & i \\ -i & -1+i \end{bmatrix}$

Solution

We have

$$\begin{aligned} CC^* &= \begin{bmatrix} 1+i & i \\ -i & -1+i \end{bmatrix}\begin{bmatrix} 1-i & i \\ -i & -1-i \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\ C^*C &= \begin{bmatrix} 1-i & i \\ -i & -1-i \end{bmatrix}\begin{bmatrix} 1+i & i \\ -i & -1+i \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \end{aligned}$$

So, C is normal.

Normal Matrices

Theorem 11.5.10

If A is a normal matrix, then

1. $\|A\vec{z}\| = \|A^*\vec{z}\|$, for all $\vec{z} \in \mathbb{C}^n$.
2. $A - \lambda I$ is normal for every $\lambda \in \mathbb{C}$.
3. If $A\vec{z} = \lambda\vec{z}$, then $A^*\vec{z} = \overline{\lambda}\vec{z}$.
4. If $\vec{z}_1$ and $\vec{z}_2$ are eigenvectors of A corresponding to distinct eigenvalues λ_1 and λ_2 of A, then $\vec{z}_1$ and $\vec{z}_2$ are orthogonal.

Proof

For 1. we have for any $\vec{z} \in \mathbb{C}^n$ that

$$\begin{aligned} \|A^*\vec{z}\|^2 \quad &= \langle A^*\vec{z}, A^*\vec{z}\rangle \\ = (A^*\vec{z})^T\overline{A^*\vec{z}} \quad &= \vec{z}^T(A^*)^T\overline{A^*}\,\overline{\vec{z}} \\ = \vec{z}^T\overline{A}\,\overline{A^*}\,\overline{\vec{z}} \quad &= \vec{z}^T\overline{AA^*}\,\overline{\vec{z}} \\ = \vec{z}^T\overline{A^*A}\,\overline{\vec{z}} \quad &= \vec{z}^T\overline{A^*}\,\overline{A}\,\overline{\vec{z}} \\ = \vec{z}^T A^T\overline{A\vec{z}} \quad &= (A\vec{z})^T\overline{A\vec{z}} \\ = \langle A\vec{z}, A\vec{z}\rangle \quad &= \|A\vec{z}\|^2 \end{aligned}$$

Normal Matrices

Theorem 11.5.10

If A is a normal matrix, then

1. $\|A\vec{z}\| = \|A^*\vec{z}\|$, for all $\vec{z} \in \mathbb{C}^n$.
2. $A - \lambda I$ is normal for every $\lambda \in \mathbb{C}$.
3. If $A\vec{z} = \lambda\vec{z}$, then $A^*\vec{z} = \overline{\lambda}\vec{z}$.
4. If $\vec{z}_1$ and $\vec{z}_2$ are eigenvectors of A corresponding to distinct eigenvalues λ_1 and λ_2 of A, then $\vec{z}_1$ and $\vec{z}_2$ are orthogonal.

Proof

For 2. we need to prove that $A - \lambda I$ is normal, and so we prove that $(A - \lambda I)(A - \lambda I)^* = (A - \lambda I)^*(A - \lambda I)$.

$$\begin{aligned}(A - \lambda I)(A - \lambda I)^* &= (A - \lambda I)(A^* - \overline{\lambda} I) \\ &= AA^* - \lambda A^* - \overline{\lambda} A + |\lambda|^2 I \\ &= A^*A - \overline{\lambda} A - \lambda A^* + |\lambda|^2 I \\ &= (A^* - \overline{\lambda} I)(A - \lambda I) \\ &= (A - \lambda I)^*(A - \lambda I)\end{aligned}$$

Normal Matrices

Theorem 11.5.10

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Proof

For 3. suppose that $A\vec{z} = \lambda\vec{z}$ for some $\vec{z} \in \mathbb{C}^n$, $\vec{z} \neq \vec{0}$ and let $B = A - \lambda I$.
Then B is normal by 2. and

$$B\vec{z} = (A - \lambda I)\vec{z} = A\vec{z} - \lambda\vec{z} = \vec{0}$$

So, by 1. we get

$$0 = \|B\vec{z}\| = \|B^*\vec{z}\| = \|(A^* - \overline{\lambda} I)\vec{z}\| = \|A^*\vec{z} - \overline{\lambda}\vec{z}\|$$

Consequently, $A^*\vec{z} - \overline{\lambda}\vec{z} = \vec{0}$ and thus, $A^*\vec{z} = \overline{\lambda}\vec{z}$, as required.

Normal Matrices

Theorem 11.5.10

If A is a normal matrix, then

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4. If $\vec{z}_1$ and $\vec{z}_2$ are eigenvectors of A corresponding to distinct eigenvalues λ_1 and λ_2 of A, then $\vec{z}_1$ and $\vec{z}_2$ are orthogonal.

Proof

The proof of 4. is left as a recommended exercise.

□

Note: Property 4. shows us that the procedure for unitarily diagonalizing a normal matrix is exactly the same as the procedure for orthogonally diagonalizing a real symmetric matrix.