Orthogonal Complements

Over the next two lectures we will generalize the concept of projections to inner product spaces.

Recall from Linear Algebra 1 that the idea of a projection of a vector $\vec{x}$ onto a plane P was to write

$$\vec{x} = \vec{u} + \vec{v}$$

where $\vec{u} \in P$ and $\vec{v}$ is orthogonal to P.
We call $\vec{u} = \text{proj}_P(\vec{x})$ the projection of $\vec{x}$ onto P and $\vec{v} = \text{perp}_P(\vec{x})$, the perpendicular of $\vec{x}$ onto P.

We now want to generalize this to the projection of a vector $\vec{x}$ onto a finite dimensional subspace $\mathbb{W}$ of an inner product space $\mathbb{V}$.

That is, we want to write $\vec{x} = \vec{u} + \vec{v}$ where $\vec{u} \in \mathbb{W}$ and $\vec{v}$ is orthogonal to $\mathbb{W}$.

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Definition: Let $\mathbb{W}$ be a subspace of an inner product space $\mathbb{V}$. The orthogonal complement $\mathbb{W}^\perp$ of $\mathbb{W}$ in $\mathbb{V}$ is defined by

$$\mathbb{W}^\perp = \{\vec{v} \in \mathbb{V} \mid \langle \vec{w}, \vec{v} \rangle = 0 \text{ for all } \vec{w} \in \mathbb{W}\}$$

Theorem 9.4.1

Let $\{\vec{v}_1, \ldots, \vec{v}_k\}$ be a basis for a subspace $\mathbb{W}$ of an inner product space $\mathbb{V}$. If $\vec{x} \in \mathbb{V}$ is orthogonal to $\vec{v}_1, \ldots, \vec{v}_k$, then $\vec{x} \in \mathbb{W}^\perp$.

Thus, to check if a vector $\vec{x}$ is in $\mathbb{W}^\perp$, we only need to check if $\vec{x}$ is orthogonal to the basis vectors.

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Example

Let $\mathbb{W} = \operatorname{Span}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \right\}$ be a subspace of $M_{2\times 2}(\mathbb{R})$. Find $\mathbb{W}^\perp$.

Solution

We want to find all matrices $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ such that

$$0 = \left\langle A, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle = a_1 + a_2 + a_3 + a_4$$

$$0 = \left\langle A, \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \right\rangle = a_1 + 2a_2 - a_3 + a_4$$

Solving the homogenous system, we get $a_1 = -3s - t$, $a_2 = 2s$, $a_3 = s$, and $a_4 = t$.

Thus, every vector in $\mathbb{W}^\perp$ has the form

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} -3s - t & 2s \\ s & t \end{bmatrix} = s\begin{bmatrix} -3 & 2 \\ 1 & 0 \end{bmatrix} + t\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

Hence, $\mathbb{W}^\perp = \operatorname{Span}\left\{ \begin{bmatrix} -3 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

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Example

Let $\mathbb{S} = \operatorname{Span}\{x\}$ in $P_2(\mathbb{R})$ with $\langle p(x), q(x) \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$. Find $\mathbb{S}^\perp$.

Solution

Let $p(x) = a + bx + cx^2 \in \mathbb{S}^\perp$.

Then, we have

$$0 = \langle a + bx + cx^2, x \rangle = a(0) + (a + b + c)(1) + (a + 2b + 4c)(2) = 3a + 5b + 9c$$

Hence, $a = -\frac{5}{3}b - 3c$.

So, we have

$$p(x) = -\frac{5}{3}b - 3c + bx + cx^2 = b\left(-\frac{5}{3} + x\right) + c(-3 + x^2), \quad b, c \in \mathbb{R}$$

Thus,

$$\mathbb{S}^\perp = \operatorname{Span}\{-5 + 3x, -3 + x^2\}$$

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Theorem 9.4.2

If $\mathbb{W}$ is a finite dimensional subspace of an inner product space $\mathbb{V}$, then

1. $\mathbb{W}^\perp$ is a subspace of $\mathbb{V}$
2. If $\dim \mathbb{V} = n$, then $\dim \mathbb{W}^\perp = n - \dim \mathbb{W}$
3. If $\mathbb{V}$ is finite dimensional, then $(\mathbb{W}^\perp)^\perp = \mathbb{W}$
4. $\mathbb{W} \cap \mathbb{W}^\perp = \{\vec{0}\}$
5. If $\dim \mathbb{V} = n$, $\{\vec{v}_1, \dots, \vec{v}_k\}$ is an orthogonal basis for $\mathbb{W}$, and $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ is an orthogonal basis for $\mathbb{W}^\perp$, then $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ is an orthogonal basis for $\mathbb{V}$

Orthogonal Complements

If $\mathbb{W}$ is a finite dimensional subspace of an inner product space $\mathbb{V}$, then

2. If $\dim \mathbb{V} = n$, then $\dim \mathbb{W}^\perp = n - \dim \mathbb{W}$

Proof of 2

Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be an orthonormal basis for $\mathbb{W}$.
Using the methods from Linear Algebra 1, we can extend this to a basis for all of $\mathbb{V}$.
We can then apply the Gram-Schmidt procedure to this basis and normalize to get an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ for $\mathbb{V}$.
We want to prove that $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ is a basis for $\mathbb{W}^\perp$.

We first observe that since the vectors $\vec{v}_{k+1}, \dots, \vec{v}_n$ are part of a basis, the set $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ must be linearly independent.

Let $\vec{x} \in \mathbb{W}^\perp$. Then $\vec{x} \in \mathbb{V}$. Therefore, we can write $\vec{x}$ as a linear combination of the orthonormal basis vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$.

$$\begin{aligned}\vec{x} &= \langle \vec{x}, \vec{v}_1\rangle \vec{v}_1 + \cdots + \langle \vec{x}, \vec{v}_k\rangle \vec{v}_k + \langle \vec{x}, \vec{v}_{k+1}\rangle \vec{v}_{k+1} + \cdots + \langle \vec{x}, \vec{v}_n\rangle \vec{v}_n \\ &= \vec{0} + \cdots + \vec{0} + \langle \vec{x}, \vec{v}_{k+1}\rangle \vec{v}_{k+1} + \cdots + \langle \vec{x}, \vec{v}_n\rangle \vec{v}_n\end{aligned}$$

Hence, $\mathbb{W}^\perp = \operatorname{Span}\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$.
Thus, $\dim \mathbb{W}^\perp = n - k = n - \dim \mathbb{W}$. ☐

Note: The proofs of all of the properties are in the course notes.