Orthonormal Bases and Orthogonal Matrices

Last Lecture

- We defined an orthogonal basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for an inner product space $\mathbb{V}$ to be a basis for $\mathbb{V}$ such that $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ for all $i \neq j$.
- We also defined an orthonormal basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for an inner product space $\mathbb{V}$ to be a basis for $\mathbb{V}$ such that $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ for all $i \neq j$, and $||\vec{v}_i|| = \sqrt{\langle \vec{v}_i, \vec{v}_i \rangle} = 1$ for $1 \leq i \leq n$.

In This Lecture

- We will show that these bases are easy to use and very useful.

Orthogonal and Orthonormal Bases

Theorem 9.2.4

If $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is an orthogonal basis for an inner product space $\mathbb{V}$ and $\vec{v} \in \mathbb{V}$, then the coefficient of $\vec{v}_i$ when $\vec{v}$ is written as a linear combination of the vectors in B is

$$\frac{\langle \vec{v}, \vec{v}_i \rangle}{||\vec{v}_i||^2}$$

In particular,

$$\vec{v} = \frac{\langle \vec{v}, \vec{v}_1 \rangle}{||\vec{v}_1||^2}\vec{v}_1 + \cdots + \frac{\langle \vec{v}, \vec{v}_n \rangle}{||\vec{v}_n||^2}\vec{v}_n$$

Proof

Consider

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

Then,

$$\begin{aligned}\langle \vec{v}, \vec{v}_i \rangle &= \langle c_1\vec{v}_1 + \cdots + c_n\vec{v}_n, \vec{v}_i \rangle \\ &= c_1\langle \vec{v}_1, \vec{v}_i \rangle + \cdots + c_n\langle \vec{v}_n, \vec{v}_i \rangle \\ &= 0 + \cdots + 0 + c_i||\vec{v}_i||^2 + 0 + \cdots + 0\end{aligned}$$

Since $\vec{v}_i \neq \vec{0}$ we get $c_i = \dfrac{\langle \vec{v}, \vec{v}_i \rangle}{||\vec{v}_i||^2}$ for $1 \leq i \leq n$.

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Orthogonal and Orthonormal Bases

Example

Verify that $B = \{1, x, 3x^2 - 2\}$ is an orthogonal basis of $P_2(\mathbb{R})$ with inner product defined by

$$\langle p(x), q(x) \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$

and find the coordinates of $1 + x + x^2$ with respect to B.

Solution

First, we have that

$$\begin{aligned} \langle 1, x \rangle &= (1)(-1) + 1(0) + 1(1) = 0 \\ \langle 1, 3x^2 - 2 \rangle &= 1(1) + 1(-2) + 1(1) = 0 \\ \langle x, 3x^2 - 2 \rangle &= (-1)(1) + 0(-2) + 1(1) = 0 \end{aligned}$$

Hence, B is an orthogonal set of 3 non-zero vectors and therefore is a linearly independent set of 3 vectors in a 3-dimensional inner product space, and so B is an orthogonal basis for $P_2(\mathbb{R})$.

Orthogonal and Orthonormal Bases

Example

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$$\langle p(x), q(x) \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$

and find the coordinates of $1 + x + x^2$ with respect to B.

Solution

To find the coordinates of $1 + x + x^2$ we use Theorem 9.2.4. We get

$$\begin{aligned} \langle 1 + x + x^2, 1 \rangle &= 1(1) + 1(1) + 3(1) = 5 \\ \langle 1 + x + x^2, x \rangle &= 1(-1) + 1(0) + 3(1) = 2 \\ \langle 1 + x + x^2, 3x^2 - 2 \rangle &= 1(1) + 1(-2) + 3(1) = 2 \\ \|1\|^2 &= \langle 1, 1 \rangle = 1^2 + 1^2 + 1^2 = 3 \\ \|x\|^2 &= \langle x, x \rangle = (-1)^2 + 0^2 + 1^2 = 2 \\ \|3x^2 - 2\|^2 &= \langle 3x^2 - 2, 3x^2 - 2 \rangle = 1^2 + (-2)^2 + 1^2 = 6 \end{aligned}$$

Thus,

$$1 + x + x^2 = \frac{5}{3}(1) + 1(x) + \frac{1}{3}(3x^2 - 2)$$

Orthogonal and Orthonormal Bases

Example

The set $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^3$. Find the coordinates of $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to B.

Solution

We get

$$\vec{x} = \langle \vec{x}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{x}, \vec{v}_2 \rangle \vec{v}_2 + \langle \vec{x}, \vec{v}_3 \rangle \vec{v}_3$$
$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0 \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} + (2\sqrt{2}) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} - \sqrt{6} \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

Orthogonal Matrices

Recall if $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ and C are both bases for a vector space $\mathbb{V}$, then the change of coordinates matrix from B-coordinates to C-coordinates is

$${}_C P_B = \begin{bmatrix} [\vec{v}_1]_C & \cdots & [\vec{v}_n]_C \end{bmatrix}$$

It satisfies $[\vec{x}]_C = {}_C P_B [\vec{x}]_B$.

In our example, the change of coordinates from the basis $B = \left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right\}$ to the standard basis $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is

$${}_S P_B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = R_\theta$$

We find that

$${}_B P_S = ({}_S P_B)^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = ({}_S P_B)^T$$

Orthogonal Matrices

Theorem 9.2.6

If $P \in M_{n\times n}(\mathbb{R})$, then the following are equivalent:

1. The columns of P form an orthonormal basis for $\mathbb{R}^n$.
2. $P^T = P^{-1}$
3. The rows of P form an orthonormal basis for $\mathbb{R}^n$.

Proof

(1) ⇔ (2): Let $P = [\vec{v}_1 \quad \cdots \quad \vec{v}_n]$. By definition of matrix multiplication we get

$$P^TP = \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_n^T \end{bmatrix} [\vec{v}_1 \quad \cdots \quad \vec{v}_n] = \begin{bmatrix} \vec{v}_1\cdot\vec{v}_1 & \vec{v}_1\cdot\vec{v}_2 & \cdots & \vec{v}_1\cdot\vec{v}_n \\ \vec{v}_2\cdot\vec{v}_1 & \vec{v}_2\cdot\vec{v}_2 & \cdots & \vec{v}_2\cdot\vec{v}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{v}_n\cdot\vec{v}_1 & \vec{v}_n\cdot\vec{v}_2 & \cdots & \vec{v}_n\cdot\vec{v}_n \end{bmatrix}$$

Therefore, $P^TP = I$ if and only if $\vec{v}_i \cdot \vec{v}_i = 1$ for all $1 \leq i \leq n$, and $\vec{v}_i \cdot \vec{v}_j = 0$ whenever $i \neq j$. That is, $P^TP = I$ if and only if $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$.

Orthogonal Matrices

Theorem 9.2.6

If $P \in M_{n\times n}(\mathbb{R})$, then the following are equivalent:

1. The columns of P form an orthonormal basis for $\mathbb{R}^n$.
2. $P^T = P^{-1}$
3. The rows of P form an orthonormal basis for $\mathbb{R}^n$.

Proof

(2) ⇔ (3):

Let $P = \begin{bmatrix} \vec{w}_1^T \\ \vdots \\ \vec{w}_n^T \end{bmatrix}$. By definition of matrix multiplication we have

$$PP^T = \begin{bmatrix} \vec{w}_1^T \\ \vdots \\ \vec{w}_n^T \end{bmatrix} [\vec{w}_1 \quad \cdots \quad \vec{w}_n] = \begin{bmatrix} \vec{w}_1\cdot\vec{w}_1 & \vec{w}_1\cdot\vec{w}_2 & \cdots & \vec{w}_1\cdot\vec{w}_n \\ \vec{w}_2\cdot\vec{w}_1 & \vec{w}_2\cdot\vec{w}_2 & \cdots & \vec{w}_2\cdot\vec{w}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{w}_n\cdot\vec{w}_1 & \vec{w}_n\cdot\vec{w}_2 & \cdots & \vec{w}_n\cdot\vec{w}_n \end{bmatrix}$$

Thus, as above, $PP^T = I$ if and only if $\{\vec{w}_1, \dots, \vec{w}_n\}$ is an orthonormal basis for $\mathbb{R}^n$.

□

Orthogonal Matrices

Definition: Let $P \in M_{n\times n}(\mathbb{R})$ whose columns form an orthonormal basis for $\mathbb{R}^n$. Then, P is called an orthogonal matrix.

Example

Which of the following are orthogonal matrices?

(a) $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}$ (c) $C = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Solution

(a) Since its columns are standard basis vectors, they form an orthonormal basis for $\mathbb{R}^3$ and hence A is orthogonal.

(b) Observe that the dot product of the first and second row is not 0, so the rows are not orthogonal and hence B is not orthogonal.

(c) We have $C^TC = I$. So, $C^T = C^{-1}$ and hence C is orthogonal.

Orthogonal Matrices

Theorem 9.2.7

If P and Q are $n \times n$ orthogonal matrices and $\vec{x}, \vec{y} \in \mathbb{R}^n$, then:

1. $(P\vec{x}) \cdot (P\vec{y}) = \vec{x} \cdot \vec{y}$
2. $||P\vec{x}|| = ||\vec{x}||$
3. $\det P = \pm 1$
4. All real eigenvalues of P are 1 or -1.
5. PQ is also an orthogonal matrix.

Proof

We will prove (1) and leave the others as exercises.

(1) We have

$$(P\vec{x}) \cdot (P\vec{y}) = (P\vec{x})^T(P\vec{y}) = \vec{x}^TP^TP\vec{y} = \vec{x}^TI\vec{y} = \vec{x}^T\vec{y} = \vec{x} \cdot \vec{y}$$

$\square$