Projections

In This Lecture

- We return to our purpose for looking at orthogonal complements: to define the projection of a vector $\vec{v}$ onto a finite dimensional subspace $\mathbb{W}$ of an inner product space $\mathbb{V}$.

Projections

Recall that we want to find $\text{proj}_{\mathbb{W}}(\vec{v})$ and $\text{perp}_{\mathbb{W}}(\vec{v})$ such that

$$\vec{v} = \text{proj}_{\mathbb{W}}(\vec{v}) + \text{perp}_{\mathbb{W}}(\vec{v})$$

with $\text{proj}_{\mathbb{W}}(\vec{v}) \in \mathbb{W}$ and $\text{perp}_{\mathbb{W}}(\vec{v}) \in \mathbb{W}^{\perp}$.

Suppose that $\dim \mathbb{V} = n$, $\{\vec{v}_1, \dots, \vec{v}_k\}$ is an orthogonal basis for $\mathbb{W}$, and $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ is an orthogonal basis for $\mathbb{W}^{\perp}$. Then $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ is an orthogonal basis for $\mathbb{V}$. Hence, for any $\vec{v} \in \mathbb{V}$ we get

$$\vec{v} = \frac{\langle \vec{v}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \cdots + \frac{\langle \vec{v}, \vec{v}_k \rangle}{\|\vec{v}_k\|^2}\vec{v}_k + \frac{\langle \vec{v}, \vec{v}_{k+1} \rangle}{\|\vec{v}_{k+1}\|^2}\vec{v}_{k+1} + \cdots + \frac{\langle \vec{v}, \vec{v}_n \rangle}{\|\vec{v}_n\|^2}\vec{v}_n$$

Definition: Suppose $\mathbb{W}$ is a k-dimensional subspace of an inner product space $\mathbb{V}$ and $\{\vec{v}_1, \dots, \vec{v}_k\}$ is an orthogonal basis for $\mathbb{W}$. For any $\vec{v} \in \mathbb{V}$ we define the projection of $\vec{v}$ onto $\mathbb{W}$ by

$$\text{proj}_{\mathbb{W}}(\vec{v}) = \frac{\langle \vec{v}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \cdots + \frac{\langle \vec{v}, \vec{v}_k \rangle}{\|\vec{v}_k\|^2}\vec{v}_k$$

and the perpendicular of $\vec{v}$ onto $\mathbb{W}$ by

$$\text{perp}_{\mathbb{W}}(\vec{v}) = \vec{v} - \text{proj}_{\mathbb{W}}(\vec{v})$$

Projections

Theorem 9.4.3

Suppose $\mathbb{W}$ is a k-dimensional subspace of an inner product space $\mathbb{V}$. For any $\vec{v} \in \mathbb{V}$, we have

$$\text{perp}_{\mathbb{W}}(\vec{v}) = \vec{v} - \text{proj}_{\mathbb{W}}(\vec{v}) \in \mathbb{W}^{\perp}$$

Proof

$$\text{perp}_{\mathbb{W}}(\vec{v}) = \vec{v} - \text{proj}_{\mathbb{W}}(\vec{v}) = \vec{v} - \frac{\langle \vec{v}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \cdots - \frac{\langle \vec{v}, \vec{v}_k \rangle}{\|\vec{v}_k\|^2}\vec{v}_k$$

By the Gram-Schmidt Orthogonalization Theorem, $\{\vec{v}_1, \ldots, \vec{v}_k, \text{perp}_{\mathbb{W}}(\vec{v})\}$ is an orthogonal set.
Therefore, $\text{perp}_{\mathbb{W}}(\vec{v}) \in \mathbb{W}^{\perp}$ by Theorem 9.4.1.

□

Note: The formula for the projection requires us to have an orthogonal or orthonormal basis. For this reason these are sometimes called orthogonal projections. Be careful when doing problems with projections that you have at least an orthogonal basis for the subspace you are projecting onto.

Projections

Theorem 9.4.4

If $\mathbb{W}$ is a k-dimensional subspace of an inner product space $\mathbb{V}$, then $\text{proj}_{\mathbb{W}}$ is a linear operator on $\mathbb{V}$ with kernel $\mathbb{W}^{\perp}$.

Theorem 9.4.5

If $\mathbb{W}$ is a subspace of a finite dimensional inner product space $\mathbb{V}$, then for any $\vec{v} \in \mathbb{V}$ we have

$$\text{proj}_{\mathbb{W}^{\perp}}(\vec{v}) = \text{perp}_{\mathbb{W}}(\vec{v})$$

Projections

Example

Let $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$ be an orthogonal basis for a subspace $\mathbb{W}$ of $\mathbb{R}^3$ and let $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.
Determine $\text{proj}_{\mathbb{W}}(\vec{x})$ and $\text{perp}_{\mathbb{W}}(\vec{x})$.

Solution

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$. Then,

$$\text{proj}_{\mathbb{W}}(\vec{x}) = \frac{\langle \vec{x}, \vec{v}_1 \rangle}{||\vec{v}_1||^2} \vec{v}_1 + \frac{\langle \vec{x}, \vec{v}_2 \rangle}{||\vec{v}_2||^2} \vec{v}_2 = \frac{7}{6} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \frac{-4}{3} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1 \\ 5/2 \end{bmatrix}$$

$$\text{perp}_{\mathbb{W}}(\vec{x}) = \vec{x} - \text{proj}_{\mathbb{W}}(\vec{x}) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 5/2 \\ 1 \\ 5/2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

Projections

Notice that if we have an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_k\}$ for $\mathbb{W}$, then the formula for the projection simplifies to

$$\text{proj}_{\mathbb{W}}(\vec{x}) = \langle \vec{x}, \vec{v}_1 \rangle \vec{v}_1 + \cdots + \langle \vec{x}, \vec{v}_k \rangle \vec{v}_k$$

Example

Let $\mathbb{S} = \text{Span}\left\{ \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} \right\}$ be a subspace of $M_{2\times2}(\mathbb{R})$ and let $A = \begin{bmatrix} 2 & 5 \\ -7 & 3 \end{bmatrix}$. Determine $\text{proj}_{\mathbb{S}}(A)$ and $\text{perp}_{\mathbb{S}}(A)$.

Solution

We can easily verify that an orthonormal basis for $\mathbb{S}$ is $B = \{B_1, B_2\} = \left\{ \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} \right\}$. Thus, we get

$$\text{proj}_{\mathbb{S}}(A) = \langle A, B_1 \rangle B_1 + \langle A, B_2 \rangle B_2 = \frac{3}{2} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + \frac{-13}{2} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} -5/2 & 4 \\ -5/2 & 4 \end{bmatrix}$$

$$\text{perp}_{\mathbb{S}}(A) = A - \text{proj}_{\mathbb{S}}(A) = \begin{bmatrix} 2 & 5 \\ -7 & 3 \end{bmatrix} - \begin{bmatrix} -5/2 & 4 \\ -5/2 & 4 \end{bmatrix} = \begin{bmatrix} 9/2 & 1 \\ -9/2 & -1 \end{bmatrix}$$

Projections

Example

Let $\mathbb{W} = \text{Span}\{1, x\}$ be a subspace of $P_2(\mathbb{R})$ under the inner product $\langle p, q\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$. Determine $\text{proj}_{\mathbb{W}}(x^2)$.

Solution

We first need to apply the Gram-Schmidt procedure to $\{1, x\}$ to find an orthogonal basis for $\mathbb{W}$.
Let $p_1(x) = 1$ and then

$$p_2(x) = x - \frac{\langle x, 1\rangle}{||1||^2} 1 = x - \frac{0(1) + 1(1) + 2(1)}{1^2 + 1^2 + 1^2} 1 = x - 1$$

Therefore, our orthogonal basis for $\mathbb{W}$ is $\{1, x - 1\}$.

$$\begin{aligned}
\text{proj}_{\mathbb{W}}(x^2) &= \frac{\langle x^2, 1\rangle}{||1||^2} 1 + \frac{\langle x^2, x-1\rangle}{||x-1||^2} (x-1) \\
&= \frac{0(1) + 1(1) + 4(1)}{1^1 + 1^1 + 1^2} 1 + \frac{0(-1) + 1(0) + 4(1)}{(-1)^2 + 0^2 + 1^2} (x-1) \\
&= \frac{5}{3} 1 + 2(x-1) \\
&= 2x - \frac{1}{3}
\end{aligned}$$