Area and Volume

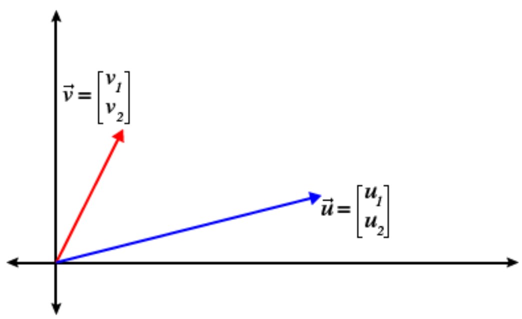
Last Lecture

- We saw that determinants can be used for more than just determining if a matrix is invertible or not.
- We used determinants in Cramer's rule to solve systems of linear equations.

In This Lecture

- We will see that the determinant has a very nice geometric interpretation.

Area of a Parallelogram



Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$.

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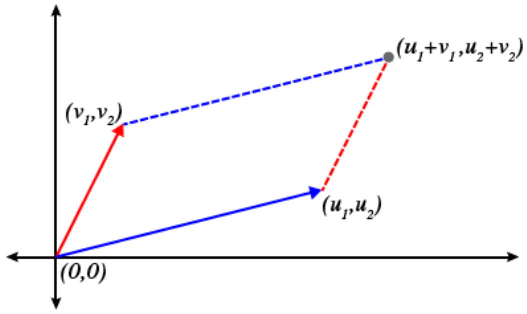

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We can then form a parallelogram in $\mathbb{R}^2$ with corner points $(0,0)$, (u_1,u_2), (v_1,v_2), and (u_1+v_1,u_2+v_2).

This is called the parallelogram induced by $\vec{u}$ and $\vec{v}$.

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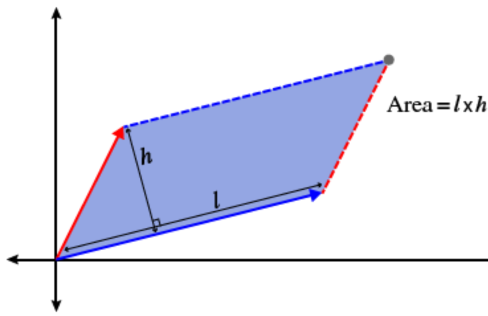

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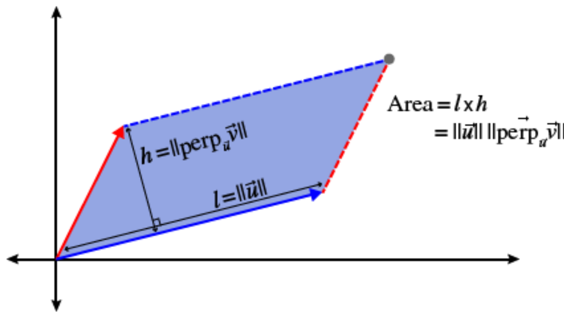

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We get

$$A = \|\vec{u}\| \|\text{perp}_{\vec{u}} \vec{v}\|$$

Using trigonometry, we see that $\|\text{perp}_{\vec{u}} \vec{v}\| = \|\vec{v}\| |\sin\theta|$.

Hence,

$$A = \|\vec{u}\| \|\vec{v}\| |\sin\theta|$$

Area of a Parallelogram

Hence,

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Recall that $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$.

Using this gives

$$\begin{aligned}
(\text{Area})^2 &= \|\vec{u}\|^2 \|\vec{v}\|^2 \sin^2\theta \\
&= \|\vec{u}\|^2 \|\vec{v}\|^2 (1 - \cos^2\theta) \\
&= \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2 \\
&= (u_1^2 + u_2^2)(v_1^2 + v_2^2) - (u_1 v_1 + u_2 v_2)^2 \\
&= u_1^2 v_2^2 + u_2^2 v_1^2 - 2(u_1 v_2 u_2 v_1) \\
&= (u_1 v_2 - u_2 v_1)^2 \\
&= \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}^2
\end{aligned}$$

Hence, the area of the parallelogram is given by $\left|\det\begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}\right|$.

Area of a Parallelogram

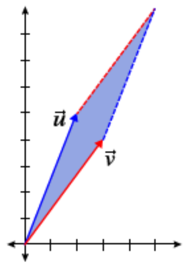
Example

Determine the area of the parallelogram induced by $\vec{u} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Solution

We have

$$A = \left| \det \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} \right| = |-7| = 7$$



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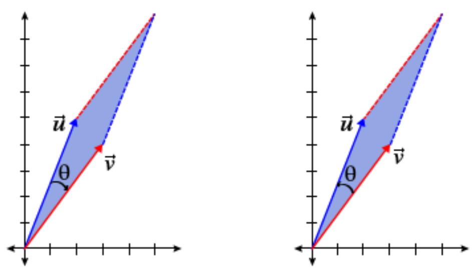
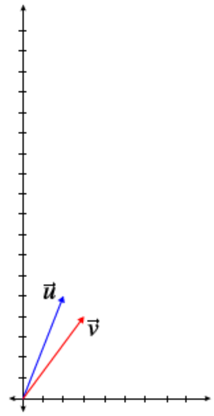


Recall that swapping columns multiplies the determinant by -1, so that means the area of the parallelogram induced by $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ is

$$A = \left| \det \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix} \right| = 7$$

Area of a Parallelogram

We can get geometric interpretations of the other row/column operations on determinants as well.



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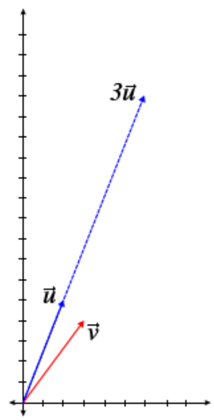

If we multiply the first column of $\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$ by 3, we get $\begin{bmatrix} 6 & 3 \\ 15 & 4 \end{bmatrix}$.

This corresponds to calculating $3\vec{u} = \begin{bmatrix} 6 \\ 15 \end{bmatrix}$.

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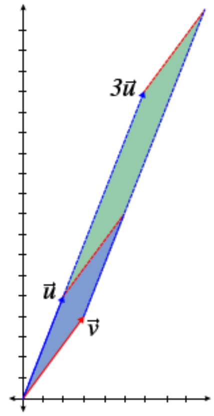

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The area of the parallelogram induced by $3\vec{u}$ and $\vec{v}$ is

$$A = \left|\det\begin{bmatrix} 6 & 3 \\ 15 & 4 \end{bmatrix}\right| = |-21| = 21$$

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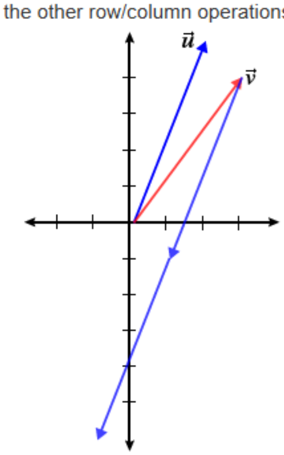

If we use the column operation $C_2 - 2C_1$ on $\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$ we get $\begin{bmatrix} 2 & -1 \\ 5 & -6 \end{bmatrix}$.

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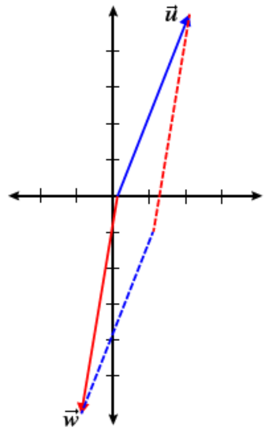

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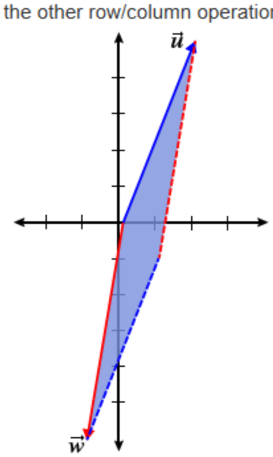

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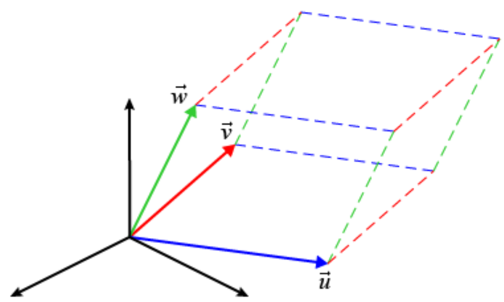
The area of this parallelogram is

$$A = \left|\det\begin{bmatrix} 2 & -1 \\ 5 & -6 \end{bmatrix}\right| = |-7| = 7$$

Volume of a Parallelepiped

Observe that three vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ in $\mathbb{R}^3$ induce a parallelepiped.

We can use the determinant to find the volume of the parallelepiped.



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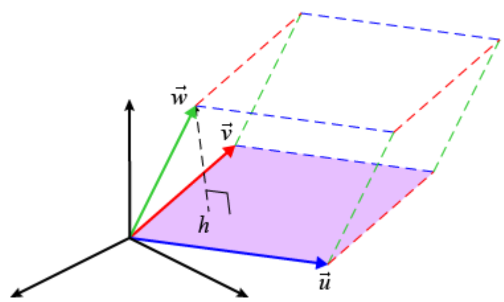
We can use the determinant to find the volume of the parallelepiped.

The volume of a parallelepiped is (area of base) $\times$ height.

The base is the parallelogram induced by vectors $\vec{u}$ and $\vec{v}$.

The area of the parallelogram is

$$\text{area of base} = \left| \|\vec{u}\| \, \|\vec{v}\| \sin\theta \right| = \|\vec{u} \times \vec{v}\|$$



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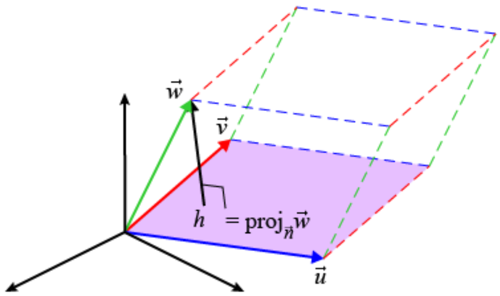

We observe that the height is the projection of $\vec{w}$ onto the normal vector $\vec{n}$ of the plane induced by $\vec{u}$ and $\vec{v}$.

Thus,

$$\text{height} = \|\text{proj}_{\vec{n}}\vec{w}\| = \left\| \frac{|\vec{w} \cdot \vec{n}|}{\|\vec{n}\|^2}\vec{n} \right\| = \frac{|\vec{w} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|\vec{w} \cdot (\vec{u} \times \vec{v})|}{\|\vec{u} \times \vec{v}\|}$$

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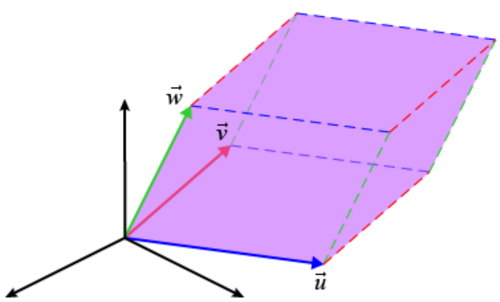

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Hence, the volume V of the parallelepiped is given by

$$\begin{aligned} V &= \frac{|\vec{w} \cdot (\vec{u} \times \vec{v})|}{\|\vec{u} \times \vec{v}\|} \times \|\vec{u} \times \vec{v}\| = |\vec{w} \cdot (\vec{u} \times \vec{v})| \\ &= |w_1(u_2v_3 - u_3v_2) - w_2(u_1v_3 - u_3v_1) + w_3(u_1v_2 - u_2v_1)| \\ &= \left| \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \right| \end{aligned}$$

Volume of a Parallelepiped

Example

Determine the volume of the parallelepiped determined by $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.

Solution

The volume is

$$V = \left|\det\begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}\right| = \left|\det\begin{bmatrix} 2 & 1 & 2 \\ -1 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}\right| = \left|\det\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix}\right| = 4$$

n-Volume of a Parallelotope

This formula works in higher dimensions as well.

If $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$, then we say that they induce an n-dimensional parallelotope.

The n-volume V of the parallelotope is

$$V = \left|\det\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}\right|$$