Properties of Vectors in $\mathbb{R}^n$ and Spanning

Last Lecture

- We defined vectors in $\mathbb{R}^n$.
- We saw how to find linear combinations of vectors.

In This Lecture

- We will look at properties of addition and scalar multiplication in $\mathbb{R}^n$.
- We will start looking at spanning and its geometric interpretation.

Properties of Vectors in $\mathbb{R}^n$

Theorem 1.1.1

If $\vec{x}, \vec{y}, \vec{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$, then

V1 $\vec{x} + \vec{y} \in \mathbb{R}^n$

V2 $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$

V3 $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

V4 There exists a vector $\vec{0} \in \mathbb{R}^n$, called the zero vector, such that $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x} \in \mathbb{R}$

V5 There exists a vector $(-\vec{x}) \in \mathbb{R}^n$ such that $\vec{x} + (-\vec{x}) = \vec{0}$

V6 $c\vec{x} \in \mathbb{R}^n$

V7 $c(d\vec{x}) = (cd)\vec{x}$

V8 $(c + d)\vec{x} = c\vec{x} + d\vec{x}$

V9 $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$

V10 $1\vec{x} = \vec{x}$

As you read over these properties,

- make sure that all of the properties make sense, and
- look for the two different types of properties.

Theorem 1.1.1
Let $\vec{x}, \vec{y}, \vec{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then:
V1 $\vec{x} + \vec{y} \in \mathbb{R}^n$;
V2 $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$;
V3 $\vec{x} + \vec{y} = \vec{y} + \vec{x}$;
V4 There exists a vector $\vec{0} \in \mathbb{R}^n$, called the zero vector, such that $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x} \in \mathbb{R}$;
V5 There exists a vector $(-\vec{x}) \in \mathbb{R}^n$ such that $\vec{x} + (-\vec{x}) = \vec{0}$;
V6 $c\vec{x} \in \mathbb{R}^n$;
V7 $c(d\vec{x}) = (cd)\vec{x}$;
V8 $(c + d)\vec{x} = c\vec{x} + d\vec{x}$;
V9 $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$;
V10 $1\vec{x} = \vec{x}$.

Proof of V1
Let $\vec{x}, \vec{y} \in \mathbb{R}^n$.

Then $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, x_i, y_i \in \mathbb{R}$ for $1 \leq i \leq n$.

By definition of addition of vectors in $\mathbb{R}^n$ we get

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

Since $x_i + y_i \in \mathbb{R}$ we get $\vec{x} + \vec{y} \in \mathbb{R}^n$ as required.

□

Theorem 1.1.1
Let $\vec{x}, \vec{y}, \vec{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then:
V1 $\vec{x} + \vec{y} \in \mathbb{R}^n$;
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V4 There exists a vector $\vec{0} \in \mathbb{R}^n$, called the zero vector, such that $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x} \in \mathbb{R}$;
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V6 $c\vec{x} \in \mathbb{R}^n$;
V7 $c(d\vec{x}) = (cd)\vec{x}$;
V8 $(c + d)\vec{x} = c\vec{x} + d\vec{x}$;
V9 $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$;
V10 $1\vec{x} = \vec{x}$.

Proof of V3
Let $\vec{x}, \vec{y} \in \mathbb{R}^n$.

Then $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, x_i, y_i \in \mathbb{R}$ for $1 \leq i \leq n$.

Observe that by definition of addition and since $x_i, y_i \in \mathbb{R}$ we get that

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} = \begin{bmatrix} y_1 + x_1 \\ \vdots \\ y_n + x_n \end{bmatrix} = \vec{y} + \vec{x}$$

as required.

□

Theorem 1.1.1
Let $\vec{x}, \vec{y}, \vec{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then:
V1 $\vec{x} + \vec{y} \in \mathbb{R}^n$;
V2 $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$;
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V4 There exists a vector $\vec{0} \in \mathbb{R}^n$, called the zero vector, such that $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x} \in \mathbb{R}$;
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V7 $c(d\vec{x}) = (cd)\vec{x}$;
V8 $(c + d)\vec{x} = c\vec{x} + d\vec{x}$;
V9 $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$;
V10 $1\vec{x} = \vec{x}$.

Proof of V4
Consider $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$.
Let $\vec{x} \in \mathbb{R}^n$.
Then $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ with $x_i \in \mathbb{R}$ for $1 \leq i \leq n$.
We get

$$\vec{x} + \vec{0} = \begin{bmatrix} x_1 + 0 \\ \vdots \\ x_n + 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{x}$$

as required. □

Theorem 1.1.1
Let $\vec{x}, \vec{y}, \vec{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then:
V1 $\vec{x} + \vec{y} \in \mathbb{R}^n$;
V2 $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$;
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V8 $(c + d)\vec{x} = c\vec{x} + d\vec{x}$;
V9 $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$;
V10 $1\vec{x} = \vec{x}$.

Remarks
V2, V3, V7, V8, V9 and V10 only refer to the operations of addition and scalar multiplication.

V1, V4, V5, and V6 are about the relationship between the operations and the set $\mathbb{R}^n$.

The proof of V4 shows the zero vector exists *and* $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

A proof of V5 gives the additive inverse of $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is

$-\vec{x} = (-1)\vec{x} = \begin{bmatrix} -x_1 \\ \vdots \\ -x_n \end{bmatrix}$

A set with property V1 is said to be closed under addition.

A set with property V6 is said to be closed under scalar multiplication.

Theorem 1.1.1 shows that $\mathbb{R}^n$ is closed under linear combinations.

That is, if $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$, then $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k \in \mathbb{R}^n$ for any $c_1, \ldots, c_k \in \mathbb{R}$.

Spanning

Definition: Let $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in $\mathbb{R}^n$. We define the span S of the set B by

$$S = \operatorname{Span} B = \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \{t_1\vec{v}_1 + \cdots + t_k\vec{v}_k \mid t_1, \dots, t_k \in \mathbb{R}\}$$

We also say that S is spanned by B and that B is a spanning set for S.

If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \mathbb{R}^n$.

Spanning

Example

Which of the following sets contain the vector $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(a) $S_1 = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Solution

We need to determine if there exists $t_1, t_2, t_3 \in \mathbb{R}$ such that

$$t_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Observe if $t_1 = 1$, $t_2 = 2$, and $t_3 = 1$, then the equation is true. Hence, $\vec{x} \in S_1$.

Spanning

Example

Which of the following sets contain the vector $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(b) $S_2 = \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Solution

We need to determine if there exists $t_1, t_2 \in \mathbb{R}$ such that

$$\begin{aligned} t_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 2t_1 \\ 0 \\ t_1 \end{bmatrix} + \begin{bmatrix} t_2 \\ 0 \\ t_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 2t_1 + t_2 \\ 0 \\ t_1 + t_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \end{aligned}$$

But, $0 \neq 2$, so these vectors cannot be equal. Hence, $\vec{x} \notin S_2$.

Viewing a Span Geometrically

Example

Describe geometrically the set $S = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

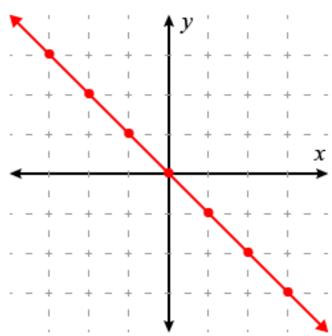
Solution

By definition, we have that

$$S = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} = \left\{ t_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mid t_1 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} t_1 \\ -t_1 \end{bmatrix} \mid t_1 \in \mathbb{R} \right\}$$

To interpret it geometrically, we can view S as the set of all points in $\mathbb{R}^2$ of the form $(t_1, -t_1)$.

Plotting these points, we that S is a line through the origin.



Vector Equation of a Set

Definition: If $S = \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$, then a vector equation for S is

$$\vec{x} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k, \qquad t_1, \dots, t_k \in \mathbb{R}$$

Example 1

A vector equation for $S = \operatorname{Span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ is

$$\vec{x} = t_1\begin{bmatrix}1\\2\\1\end{bmatrix} + t_2\begin{bmatrix}1\\0\\1\end{bmatrix}, \qquad t_1, t_2 \in \mathbb{R}$$

Vector Equation of a Set

Example 2

If T has vector equation

$$\vec{x} = s_1\begin{bmatrix}1\\0\end{bmatrix} + s_2\begin{bmatrix}0\\1\end{bmatrix} + s_3\begin{bmatrix}2\\-1\end{bmatrix}, \qquad s_1, s_2, s_3 \in \mathbb{R}$$

then we may write

$$T = \operatorname{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\-1\end{bmatrix}\right\}$$

$$\begin{aligned}\vec{x} &= s_1\begin{bmatrix}1\\0\end{bmatrix} + s_2\begin{bmatrix}0\\1\end{bmatrix} + s_3\left(2\begin{bmatrix}1\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\1\end{bmatrix}\right)\\ &= (s_1 + 2s_3)\begin{bmatrix}1\\0\end{bmatrix} + (s_2 - s_3)\begin{bmatrix}0\\1\end{bmatrix}\\ &= t_1\begin{bmatrix}1\\0\end{bmatrix} + t_2\begin{bmatrix}0\\1\end{bmatrix}\end{aligned}$$

where $t_1 = s_1 + 2s_3$ and $t_2 = s_2 - s_3$ can take any real values.

This is a simplified vector equation for T.

Also, we get that

$$T = \operatorname{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$$

Simplifying a Spanning Set

Theorem 1.1.2

If $\vec{v}_k$ can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_{k-1}$, then

$$\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$$

If $\vec{v}_k$ can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_{k-1}$, then $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$.

Proof

Process

1. Identify what assumptions you are making.
2. Identify what you are trying to prove.
3. Identify how you are going to try to prove it.

We are assuming that $\vec{v}_k$ can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_{k-1}$.
We are assuming that there exist $c_1, \dots, c_{k-1} \in \mathbb{R}$ such that

$$c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1} = \vec{v}_k$$

If $\vec{v}_k$ can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_{k-1}$, then $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$.

Proof

We are assuming that there exist $c_1, \dots, c_{k-1} \in \mathbb{R}$ such that

$$c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1} = \vec{v}_k$$

Process

1. Identify what assumptions you are making.
2. Identify what you are trying to prove.
3. Identify how you are going to try to prove it.

We want to prove that $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$.
How can we show that two sets are equal?
We will need to show that $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$ and then $\text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\} \subseteq \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$

How do we show that $A \subseteq B$?
We show that every element $a \in A$ belongs to the set B.

Let $\vec{x} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$.
We want to show that $\vec{x} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$.
We want to write $\vec{x}$ as a linear combination of the vectors $\vec{v}_1, \dots, \vec{v}_{k-1}$.

If $\vec{v}_k$ can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_{k-1}$, then $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$.

Proof

We are assuming that there exist $c_1, \dots, c_{k-1} \in \mathbb{R}$ such that

$$c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1} = \vec{v}_k$$

Let $\vec{x} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$.

Then there exist $d_1, \dots, d_k \in \mathbb{R}$ such that

$$\begin{aligned}\vec{x} &= d_1\vec{v}_1 + \cdots + d_{k-1}\vec{v}_{k-1} + d_k\vec{v}_k \\ &= d_1\vec{v}_1 + \cdots + d_{k-1}\vec{v}_{k-1} \\ &\quad + d_k(c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1}) \\ &= (d_1 + d_kc_1)\vec{v}_1 + \cdots \\ &\quad + (d_{k-1} + d_kc_{k-1})\vec{v}_{k-1}\end{aligned}$$

Thus, $\vec{x} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$.

Hence, $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$.

Process

We will need to show that $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$ and then $\text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\} \subseteq \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$

By definition, if $x \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ then there exist $d_1, \dots, d_k \in \mathbb{R}$ such that $\vec{x} = d_1\vec{v}_1 + \cdots + d_k\vec{v}_k$.
We want to remove $\vec{v}_k$ from the linear combination.
Recall that our initial assumption is that $\vec{v}_k$ is a linear combination of $\vec{v}_1, \dots, \vec{v}_{k-1}$.

If $\vec{v}_k$ can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_{k-1}$, then $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$.

Proof

If $\vec{y} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$ then there exist $t_1, \dots, t_{k-1} \in \mathbb{R}$ such that $\vec{y} = t_1\vec{v}_1 + \cdots + t_{k-1}\vec{v}_{k-1}$.

We can rewrite this as $\vec{y} = t_1\vec{v}_1 + \cdots + t_{k-1}\vec{v}_{k-1} + 0\vec{v}_k$.

So $\vec{y}$ is a linear combinations of $\vec{v}_1, \dots, \vec{v}_k$, and $\text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\} \subseteq \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$.

Therefore $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$. $\square$

Process

We will need to show that $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$ and then $\text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\} \subseteq \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$

If $\vec{v}_k$ can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_{k-1}$, then $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$.

Proof

If $\vec{y} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$ then there exist $t_1, \dots, t_{k-1} \in \mathbb{R}$ such that $\vec{y} = t_1\vec{v}_1 + \cdots + t_{k-1}\vec{v}_{k-1}$.

We can rewrite this as $\vec{y} = t_1\vec{v}_1 + \cdots + t_{k-1}\vec{v}_{k-1} + 0\vec{v}_k$.

So $\vec{y}$ is a linear combinations of $\vec{v}_1, \dots, \vec{v}_k$, and $\text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\} \subseteq \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$.

Therefore $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$. $\square$

OR

Clearly, we have $\text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\} \subseteq \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$.

Therefore $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$. $\square$

Notes

1. Figuring out the proof involved asking a lot of questions and applying a lot of definitions and theorems.
2. Notice that it was easy to show $\text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\} \subseteq \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$.

If $\vec{v}_k$ can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_{k-1}$, then $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$.

Proof

We are assuming that there exist $c_1, \dots, c_{k-1} \in \mathbb{R}$ such that

$$c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1} = \vec{v}_k$$

Let $\vec{x} \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$. Then there exist $d_1, \dots, d_k \in \mathbb{R}$ such that

$$\begin{aligned}\vec{x} &= d_1\vec{v}_1 + \cdots + d_{k-1}\vec{v}_{k-1} + d_k\vec{v}_k \\ &= d_1\vec{v}_1 + \cdots + d_{k-1}\vec{v}_{k-1} + d_k(c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1}) \\ &= (d_1 + d_k c_1)\vec{v}_1 + \cdots + (d_{k-1} + d_k c_{k-1})\vec{v}_{k-1}\end{aligned}$$

Thus, $\vec{x} \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$. Hence, $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$.

Clearly, we have $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\} \subseteq \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ and so

$$\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$$

as required. $\square$

Note that by rearranging the vectors in the set we see that Theorem 1.1.2 shows that any vector that can be written as a linear combination of the other vectors can be removed from the set without changing the set it spans.

Simplifying a Spanning Set

Example

Write a simplified vector equation for the set S spanned by $\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\4\\2\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$.

Solution

By definition, we have

$$S = \operatorname{Span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\4\\2\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$

Observe that $\begin{bmatrix}2\\4\\2\end{bmatrix} = 2\begin{bmatrix}1\\2\\1\end{bmatrix} + 0\begin{bmatrix}0\\0\\0\end{bmatrix}$.

Thus, $\begin{bmatrix}2\\4\\2\end{bmatrix}$ is a linear combination of the other vectors.

So by Theorem 1.1.2 we get that

$$S = \operatorname{Span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$

Simplifying a Spanning Set

Example

Write a simplified vector equation for the set S spanned by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Solution

$$S = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Also, clearly $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Hence, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

So applying Theorem 1.1.2 again gives

$$S = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Clearly the spanning set of S cannot be simplified any further, so a simplified vector equation for S is

$$\vec{x} = t\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

Simplifying a Spanning Set

Example

Find a simplified spanning set for the set T with vector equation

$$\vec{x} = c_1\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad c_1, c_2, c_3 \in \mathbb{R}$$

Solution

By definition, we have that

$$T = \operatorname{Span}\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Observe that

$$\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence, we can apply Theorem 1.1.2 to get that

$$T = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Observe that the spanning set cannot be simplified further since neither vector in the spanning set can be written as a linear combination of the other.

Hence, a simplified spanning set for T is $\left\{ \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Simplifying a Spanning Set

Example

Find a simplified spanning set for the set T with vector equation

$$\vec{x} = c_1\begin{bmatrix}3\\0\\1\end{bmatrix} + c_2\begin{bmatrix}0\\3\\2\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\end{bmatrix} \qquad c_1, c_2, c_3 \in \mathbb{R}$$

Solution

Another simplified spanning set for T is $\left\{\begin{bmatrix}3\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

since

$$\begin{bmatrix}0\\3\\2\end{bmatrix} = -\begin{bmatrix}3\\0\\1\end{bmatrix} + 3\begin{bmatrix}1\\1\\1\end{bmatrix}$$