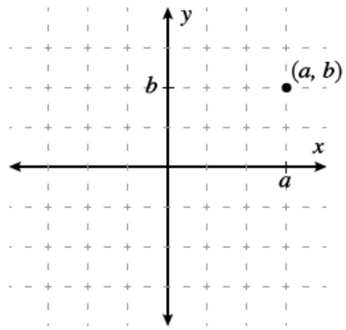
Vector Addition and Scalar Multiplication

Recall that 2-dimensional Euclidean space, $\mathbb{R}^2$, is the set of all points (x, y) where $x, y \in \mathbb{R}$.

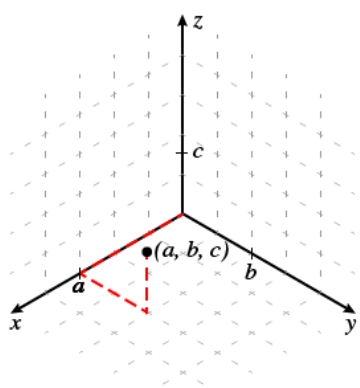
$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$$



Vector Addition and Scalar Multiplication

Similarly, 3 dimensional Euclidean space, $\mathbb{R}^3$, is the set of all points (x, y, z) where $x, y, z \in \mathbb{R}$.

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

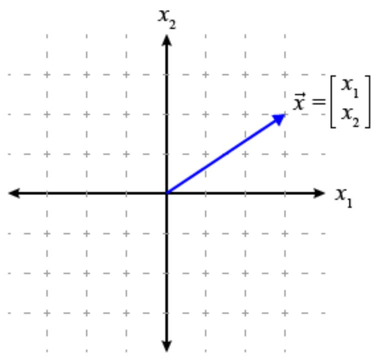


Vectors

Instead of viewing the elements of $\mathbb{R}^2$ and $\mathbb{R}^3$ as points, we will view them as objects called vectors which we can perform operations on.
We will denote vectors with the vector symbol $\vec{}$ and will write the components of a vector in a column to distinguish it from a point.

For Example

In $\mathbb{R}^2$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



In set notation,

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \right\}$$

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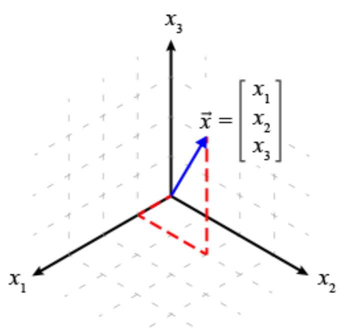

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$$\mathbb{R}^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}$$

Geometric Interpretation

We sometimes wish to get a geometric interpretation of a set of vectors.
We will do this by identifying a vector with its corresponding point.

For Example

- In $\mathbb{R}^2$, interpret $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ as the point (x_1, x_2)
- $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ represents the origin
- $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ represents the point $(-1, 3)$

Geometric Interpretation

Example

Find a geometric interpretation of $S = \left\{ \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} \mid x_1 \in \mathbb{R} \right\}$.

Solution

We begin by finding some vectors in S and plotting the corresponding points.

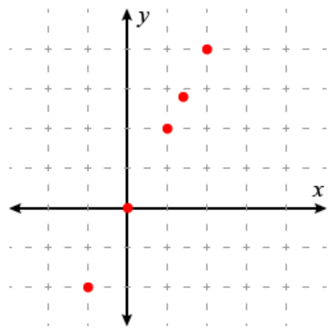
Taking $x_1 = 1$ gives $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in S$, so we plot the point $(1, 2)$.

Taking $x_1 = 2$ gives $\begin{bmatrix} 2 \\ 4 \end{bmatrix} \in S$, so we plot the point $(2, 4)$.

Taking $x_1 = 0$ gives $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in S$, so we plot the point $(0, 0)$.

Taking $x_1 = -1$ gives $\begin{bmatrix} -1 \\ -2 \end{bmatrix} \in S$, so we plot the point $(-1, -2)$.

Taking $x_1 = \sqrt{2}$ gives $\begin{bmatrix} \sqrt{2} \\ 2\sqrt{2} \end{bmatrix} \in S$, so we plot the point $(\sqrt{2}, 2\sqrt{2})$.



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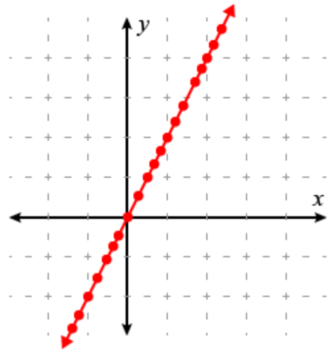

Vectors in $\mathbb{R}^n$

Definition: For any positive integer n we define n-dimensional Euclidean space $\mathbb{R}^n$ by

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mid x_1, \dots, x_n \in \mathbb{R} \right\}$$

If $\vec{x}, \vec{y} \in \mathbb{R}^n$, then there exists $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$ such that

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Definition: If $\vec{x}, \vec{y} \in \mathbb{R}^n$ then we write $\vec{x} = \vec{y}$ if and only if $x_i = y_i$ for $1 \le i \le n$.

Operations on $\mathbb{R}^n$

Definition: Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$. We define addition by

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

Definition Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $c \in \mathbb{R}$. The scalar multiplication of $\vec{x}$ by c is defined by

$$c\vec{x} = c\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix}$$

Example 1

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 1+4 \\ 2+(-3) \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

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Example 2

$$\sqrt{3}\begin{bmatrix} 1 \\ -2 \\ \pi \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ -2\sqrt{3} \\ \pi\sqrt{3} \end{bmatrix}$$

Operations on $\mathbb{R}^n$

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Example 3

$$2\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + (-2)\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -6 \\ -2 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Linear Combinations

Definition: Let $\vec{v}_1, \dots, \vec{v}_k$ be vectors in $\mathbb{R}^n$. We call a sum of scalar multiples of these vectors a linear combination. That is, a linear combination of $\vec{v}_1, \dots, \vec{v}_k$ is

$$t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k$$

where $t_1, t_2, \dots, t_k$ are real constants.

Definition: (subtraction of vectors) For any $\vec{x}, \vec{y} \in \mathbb{R}^n$, by $\vec{x} - \vec{y}$ we mean $\vec{x} + (-1)\vec{y}$.