Vector Spaces

In This Lecture

- We will define the abstract concept of real vector spaces.
- We will show a variety of examples of vector spaces.

Vector Spaces

Definition: A real vector space is a set $\mathbb{V}$ together with an operation of addition, denoted $\vec{x}+\vec{y}$, and an operation of scalar multiplication, denoted $t\vec{x}$ for any $t \in \mathbb{R}$, such that for any $\vec{x}, \vec{y}, \vec{z} \in \mathbb{V}$ and $a, b \in \mathbb{R}$ we have all of the following properties:

V1 $\vec{x}+\vec{y} \in \mathbb{V}$
V2 $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$
V3 $\vec{x}+\vec{y} = \vec{y}+\vec{x}$
V4 There exists a vector $\vec{0} \in \mathbb{V}$, called the zero vector, such that $\vec{x}+\vec{0} = \vec{x}$ for all $\vec{x} \in \mathbb{V}$
V5 There exists a vector $(-\vec{x}) \in \mathbb{V}$ such that $\vec{x}+(-\vec{x}) = \vec{0}$
V6 $a\vec{x} \in \mathbb{V}$
V7 $a(b\vec{x}) = (ab)\vec{x}$
V8 $(a+b)\vec{x} = a\vec{x}+b\vec{x}$
V9 $a(\vec{x}+\vec{y}) = a\vec{x}+a\vec{y}$
V10 $1\vec{x} = \vec{x}$

Whenever we say 'vector space' in this course, we mean 'real vector space'.

Notes

1. We sometimes denote addition in a vector space by $\oplus$ and scalar multiplication by $\odot$ to stress that the operations are non-standard.
2. In some cases we will denote the zero vector of a vector space $\mathbb{V}$ by $\vec{0}_{\mathbb{V}}$ to stress which vector space it belongs to.

Vector Spaces

Example 1

$\mathbb{R}^n$ is a vector space with addition and scalar multiplication defined in the usual way.
We call these the standard addition and scalar multiplication.
We have seen that $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ and the additive inverse of $\vec{x}$ is $(-\vec{x}) = (-1)\vec{x}$.

Example 2

$M_{m \times n}(\mathbb{R})$, the set of all $m \times n$ matrices, is a vector space with standard addition and scalar multiplication of matrices.
We have seen that $\vec{0}$ is the zero matrix and the additive inverse of A is $(-A) = (-1)A$.

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Example 3

The set $\mathbb{L}$ of all linear mappings $L : \mathbb{R}^n \to \mathbb{R}^m$ with standard addition and scalar multiplication of linear mappings is a vector space.
The zero vector is the linear mapping $0 : \mathbb{R}^n \to \mathbb{R}^m$ such that $0(x) = \vec{0}$ and the additive inverse of L is $(-L) = (-1)L$.

Example 4

The set $P_n(\mathbb{R})$ of all polynomials of degree at most n with real coefficients is a vector space with standard addition and scalar multiplication of polynomials.
The zero vector in $P_n(\mathbb{R})$ is the zero polynomial $z(x) = 0 = 0 + 0x + \cdots + 0x^n$ and the additive inverse of $p(x) = a_0 + a_1x + \cdots + a_nx^n$ is $(-p)(x) = (-1)p(x) = -a_0 - a_1x - \cdots - a_nx^n$.

Notes

- Notice in all of the examples of vector spaces above that we have that the scalar 0 times any vector is the zero vector and that (-1) times any vector $\vec{v}$ is the additive inverse of $\vec{v}$.
- Since all vector spaces have the same structure, these properties should hold for all vector spaces.

Vector Spaces

Theorem 4.1.1

If $\mathbb{V}$ is a vector space and $\vec{v} \in \mathbb{V}$, then

1. $\vec{0} = 0\vec{v}$
2. $(-\vec{v}) = (-1)\vec{v}$

Proof of $\vec{0} = 0\vec{v}$

For all $\vec{v} \in \mathbb{V}$ we have

$$\begin{aligned}
0\vec{v} &= 0\vec{v} + \vec{0} && \text{by V4} \\
&= 0\vec{v} + [\vec{v} + (-\vec{v})] && \text{by V5} \\
&= 0\vec{v} + [1\vec{v} + (-\vec{v})] && \text{by V10} \\
&= [0\vec{v} + 1\vec{v}] + (-\vec{v}) && \text{by V2} \\
&= (0 + 1)\vec{v} + (-\vec{v}) && \text{by V8} \\
&= 1\vec{v} + (-\vec{v}) && \text{operation of numbers in } \mathbb{R} \\
&= \vec{v} + (-\vec{v}) && \text{by V10} \\
&= \vec{0} && \text{by V5}
\end{aligned}$$

□

Vector Spaces

Theorem 4.1.1

If $\mathbb{V}$ is a vector space and $\vec{v} \in \mathbb{V}$, then

1. $\vec{0} = 0\vec{v}$
2. $(-\vec{v}) = (-1)\vec{v}$

This theorem shows that we can find the zero vector in any vector space by picking any vector $\vec{v}$ in the vector space and multiplying it by the scalar 0.

Notice that this implies that the zero vector in a vector space is unique.

The theorem also demonstrates that the additive inverse of any vector $\vec{v}$ in the vector space is unique and equals to $(-1)\vec{v}$.

Vector Spaces

Example

Let $\mathbb{D} = \{x \in \mathbb{R} \mid x > 0\}$ and define addition by $x \oplus y = xy$ and scalar multiplication by $t \odot x = x^t$. We will prove that $\mathbb{D}$ with these operations is a vector space.

Solution

Pick $x, y, z \in \mathbb{D}$ and $c, d \in \mathbb{R}$. Since $x, y, z \in \mathbb{D}$, we have that $x > 0$, $y > 0$, and $z > 0$.

V1 $x \oplus y = xy > 0$ since $x > 0$ and $y > 0$. Hence, $x \oplus y \in \mathbb{D}$.

V2 $(x \oplus y) \oplus z = (xy) \oplus z = (xy)z = x(yz) = x \oplus (yz) = x \oplus (y \oplus z)$.

V3 $x \oplus y = xy = yx = y \oplus x$.

V4 By Theorem 4.1.1, if $\mathbb{D}$ is a vector space, then the zero vector must be $0 \odot x = x^0 = 1$.
Indeed we see that $1 \in \mathbb{D}$ and it satisfies $x \oplus 1 = x1 = x$. Thus $\vec{0} = 1$.

V5 By Theorem 4.1.1, if $\mathbb{D}$ is a vector space, then the additive inverse of any $x \in \mathbb{D}$ must be $(-1) \odot x = x^{-1} = \frac{1}{x}$.
Observe that $\frac{1}{x} > 0$ since $x > 0$, hence $(-1) \odot x \in \mathbb{D}$. Also, $\frac{1}{x} \oplus x = \frac{1}{x} x = 1$.
Hence $\frac{1}{x}$ is the additive inverse of x.

V6 $c \odot x = x^c > 0$ since $x > 0$. Hence $c \odot x \in \mathbb{D}$.

V7 $c \odot (d \odot x) = c \odot x^d = (x^d)^c = x^{cd} = (cd) \odot x$.

V8 $(c + d) \odot x = x^{c+d} = x^c x^d = x^c \oplus x^d = (c \odot x) \oplus (d \odot x)$.

V9 $c \odot (x \oplus y) = c \odot (xy) = (xy)^c = x^c y^c = x^c \oplus y^c = (c \odot x) \oplus (c \odot y)$.

V10 $1 \odot x = x^1 = x$.

Observe that we can relate numbers in $\mathbb{R}$ to numbers in $\mathbb{D}$ by any number $x \in \mathbb{R}$ with the number $2^x \in \mathbb{D}$.
That is $0 \in \mathbb{R}$ corresponds to $2^0 = 1$ in $\mathbb{D}$, and $1 + 2 = 3$ in $\mathbb{R}$ corresponds to $2 \oplus 4 = 8$ in $\mathbb{D}$.