Vector Spaces and Subspaces

Last Lecture

- We introduced the concept of an abstract vector space.
- We saw a vector space was defined to be a set with two operations of addition and scalar multiplication that satisfy the same 10 properties as vectors in $\mathbb{R}^n$.

Definition: A real vector space is a set $\mathbb{V}$ together with an operation of addition, denoted $\vec{x}+\vec{y}$, and an operation of scalar multiplication, denoted $t\vec{x}$ for any $t \in \mathbb{R}$, such that for any $\vec{x}, \vec{y}, \vec{z} \in \mathbb{V}$ and $a, b \in \mathbb{R}$ we have all of the following properties:

V1 $\vec{x}+\vec{y} \in \mathbb{V}$
V2 $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$
V3 $\vec{x}+\vec{y} = \vec{y}+\vec{x}$
V4 There exists a vector $\vec{0} \in \mathbb{V}$, called the zero vector, such that $\vec{x}+\vec{0} = \vec{x}$
V5 There exists a vector $-\vec{x} \in \mathbb{V}$ such that $\vec{x}+(-\vec{x}) = \vec{0}$
V6 $a\vec{x} \in \mathbb{V}$
V7 $a(b\vec{x}) = (ab)\vec{x}$
V8 $(a+b)\vec{x} = a\vec{x}+b\vec{x}$
V9 $a(\vec{x}+\vec{y}) = a\vec{x}+a\vec{y}$
V10 $1\vec{x} = \vec{x}$

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In This Lecture

- We will look at examples of determining if a set with two operations is a vector space or not.
- We will then look at spaces which are subsets of other spaces.

Vector Spaces

Example 1

Is the empty set a vector space?

Solution

No! By V4, a vector space must contain at least one vector, the zero vector.

Vector Spaces

Example 2

Let $\mathbb{V} = \{(x, y) \mid x, y \in \mathbb{R}\}$ with addition defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (2x_1 + x_2, y_1 + 2y_2)$$

and scalar multiplication defined by

$$t \odot (x_1, y_1) = (tx_1, ty_1)$$

for any $t \in \mathbb{R}$. Is $\mathbb{V}$ a vector space under these operations?

Solution

Observe that

$$(1, 2) \oplus (3, 5) = \big(2(1) + 3, 2 + 2(5)\big) = (5, 12)$$

but

$$(3, 5) \oplus (1, 2) = \big(2(3) + 1, 5 + 2(2)\big) = (7, 9)$$

so, $(1, 2) \oplus (3, 5) \neq (3, 5) \oplus (1, 2)$.
Therefore, V3 does not hold, so $\mathbb{V}$ is not a vector space with these operations.

Vector Spaces

Example 3

Let $\mathbb{S} = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mid a_1 + a_2 + a_3 + a_4 = 0, a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}$. Is $\mathbb{S}$ a vector space under standard addition and scalar multiplication of matrices?

Solution

Recall

Theorem 1.1.1

Let $\vec{x}, \vec{y}, \vec{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then:

V1 $\vec{x} + \vec{y} \in \mathbb{R}^n$;

V2 $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$;

V3 $\vec{x} + \vec{y} = \vec{y} + \vec{x}$;

V4 There exists a vector $\vec{0} \in \mathbb{R}^n$, called the zero vector, such that $\vec{x} + \vec{0} = \vec{x}$;

V5 There exists a vector $-\vec{x} \in \mathbb{R}^n$ such that $\vec{x} + (-\vec{x}) = \vec{0}$;

V6 $c\vec{x} \in \mathbb{R}^n$;

V7 $c(d\vec{x}) = (cd)\vec{x}$;

V8 $(c + d)\vec{x} = c\vec{x} + d\vec{x}$;

V9 $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$;

V10 $1\vec{x} = \vec{x}$.

Vector Spaces

Example 3

Let $\mathbb{S} = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mid a_1 + a_2 + a_3 + a_4 = 0, a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}$. Is $\mathbb{S}$ a vector space under standard addition and scalar multiplication of matrices?

Solution

Recall

Theorem 3.1.1

For all $A, B, C \in M_{m\times n}(\mathbb{R})$ and $s, t \in \mathbb{R}$ we have

V1 $A + B \in M_{m\times n}(\mathbb{R})$

V2 $(A + B) + C = A + (B + C)$

V3 $A + B = B + A$

V4 There exists a matrix, denoted by $O_{m,n}$, such that $A + O_{m,n} = A$. In particular, $O_{m,n}$ is the $m \times n$ matrix with all entries zero and is called the zero matrix.

V5 There exists an $m \times n$ matrix $(-A)$, with the property that $A + (-A) = O_{m,n}$. $(-A)$ is called the addititive inverse of A.

V6 $sA \in M_{m\times n}(\mathbb{R})$

V7 $s(tA) = (st)A$

V8 $(s + t)A = sA + tA$

V9 $s(A + B) = sA + sB$

V10 $1A = A$

Vector Spaces

Example 3

Let $\mathbb{S} = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mid a_1 + a_2 + a_3 + a_4 = 0, a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}$. Is $\mathbb{S}$ a vector space under standard addition and scalar multiplication of matrices?

Solution

Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \in \mathbb{S}$.
Then $a_1 + a_2 + a_3 + a_4 = 0$ and $b_1 + b_2 + b_3 + b_4 = 0$.

Vector Spaces

Example 3

Let $\mathbb{S} = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mid a_1 + a_2 + a_3 + a_4 = 0, a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}$. Is $\mathbb{S}$ a vector space under standard addition and scalar multiplication of matrices?

Solution

Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \in \mathbb{S}$.
Then $a_1 + a_2 + a_3 + a_4 = 0$ and $b_1 + b_2 + b_3 + b_4 = 0$.

V1 - Observe that

$$A + B = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix}$$

and

$$(a_1 + b_1) + \cdots + (a_4 + b_4) = (a_1 + a_2 + a_3 + a_4) + (b_1 + b_2 + b_3 + b_4) = 0 + 0 = 0$$

Hence, $A + B \in \mathbb{S}$.

Vector Spaces

Example 3

Let $\mathbb{S} = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mid a_1 + a_2 + a_3 + a_4 = 0, a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}$. Is $\mathbb{S}$ a vector space under standard addition and scalar multiplication of matrices?

Solution

Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \in \mathbb{S}$.
Then $a_1 + a_2 + a_3 + a_4 = 0$ and $b_1 + b_2 + b_3 + b_4 = 0$.

V4 - Since we are using the same operations as $M_{2\times2}(\mathbb{R})$, the zero vector of $\mathbb{S}$ must be the zero vector of $M_{2\times2}(\mathbb{R})$. That is,

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

We see that $O_{2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{S}$ since $0 + 0 + 0 + 0 = 0$.

Vector Spaces

Example 3

Let $\mathbb{S} = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mid a_1 + a_2 + a_3 + a_4 = 0, a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}$. Is $\mathbb{S}$ a vector space under standard addition and scalar multiplication of matrices?

Solution

Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \in \mathbb{S}$.
Then $a_1 + a_2 + a_3 + a_4 = 0$ and $b_1 + b_2 + b_3 + b_4 = 0$.

V5 - If $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in S$, then $-A = \begin{bmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{bmatrix}$ since $A + (-A) = O_{2,2}$.
$-A \in S$ since

$$(-a_1) + (-a_2) + (-a_3) + (-a_4) = (-1)(a_1 + a_2 + a_3 + a_4) = (-1)(0) = 0$$

V6 - For any $t \in \mathbb{R}$, we have $tA = \begin{bmatrix} ta_1 & ta_2 \\ ta_3 & ta_4 \end{bmatrix} \in \mathbb{S}$ since

$$ta_1 + ta_2 + ta_3 + ta_4 = t(a_1 + a_2 + a_3 + a_4) = t(0) = 0$$

Therefore, $\mathbb{S}$ is a vector space.

Subspaces

Definition: If $\mathbb{S}$ is a non-empty subset of a vector space $\mathbb{V}$, and $\mathbb{S}$ is also a vector space using the same operations as $\mathbb{V}$, then $\mathbb{S}$ is called a subspace of $\mathbb{V}$.

Theorem 4.1.2 - Subspace Test

A non-empty subset $\mathbb{S}$ of a vector space $\mathbb{V}$ is a subspace of $\mathbb{V}$ if for all $\vec{x}, \vec{y} \in \mathbb{S}$ and $t \in \mathbb{R}$ we have
V1 $\vec{x} + \vec{y} \in \mathbb{S}$ (closed under addition)
V6 $t\vec{x} \in \mathbb{S}$ (closed under scalar multiplication)
under the operations of $\mathbb{V}$.

Notes

1. The proof of the Subspace Test is essentially the same as the proof for the Subspace Test in $\mathbb{R}^n$.
2. It is important not to forget to show that $\mathbb{S}$ is a non-empty subset of $\mathbb{V}$. As with subspaces of $\mathbb{R}^n$, the best way to show the set is non-empty is to show that $\vec{0}_{\mathbb{V}} \in \mathbb{S}$.

Subspaces

Example 1

Let $\mathbb{V}$ be a vector space. Show that $\mathbb{V}$ and $\mathbb{U} = \{\vec{0}_{\mathbb{V}}\}$ are both subspaces of $\mathbb{V}$.

Solution

Notice that $\mathbb{V}$ is a subset of itself and is a vector space under its own operations, and so it satisfies the defintion of a subspace. Therefore, every vector space is a subspace of itself.
Clearly, $\mathbb{U}$ is a non-empty subset of $\mathbb{V}$. So, we pick any two vectors $\vec{x}, \vec{y} \in \mathbb{U}$.
Therefore, $\vec{x} = \vec{0}_{\mathbb{V}}$ and $\vec{y} = \vec{0}_{\mathbb{V}}$.

V1 - We have $\vec{x} + \vec{y} = \vec{0}_{\mathbb{V}} + \vec{0}_{\mathbb{V}} = \vec{0}_{\mathbb{V}}$ by property V4 since $\mathbb{V}$ is a vector space.
Hence, $\vec{x} + \vec{y} \in \mathbb{U}$.

V6 - We have

$$\begin{aligned} t\vec{x} &= t\vec{0}_{\mathbb{V}} \\ &= t(0\vec{x}) && \text{by Theorem 4.1.1 since } \mathbb{V} \text{ is a vector space} \\ &= (t(0))\vec{x} && \text{by V7 since } \mathbb{V} \text{ is a vector space} \\ &= 0\vec{x} && \text{by normal operations on real numbers} \\ &= \vec{0} && \text{by Theorem 4.1.1 since } \mathbb{V} \text{ is a vector space} \end{aligned}$$

Thus, $t\vec{x} \in \mathbb{U}$.
Hence, by the Subspace Test, $\{\vec{0}_{\mathbb{V}}\}$ is a subspace of $\mathbb{V}$.

Subspaces

Example 2

Let $\mathbb{W} = \{p(x) \in P_3(\mathbb{R}) \mid p(1) = 0\}$. Show $\mathbb{W}$ is a vector space under standard addition and scalar multiplication of polynomials.

Solution

We will prove that $\mathbb{W}$ is a vector space, by showing that it is a subspace of $P_3(\mathbb{R})$.
By definition, $\mathbb{W}$ is a subset of $P_3(\mathbb{R})$.
Also, the zero vector of $P_3(\mathbb{R})$ is the zero polynomial $z(x) = 0$ which satisfies $z(1) = 0$, so $\vec{0} \in \mathbb{W}$.
Hence, $\mathbb{W}$ is non-empty.
Let $p(x), q(x) \in \mathbb{W}$. Then, $p(1) = 0$ and $q(1) = 0$.
V1 - We have $(p+q)(1) = p(1) + q(1) = 0 + 0 = 0$ so $(p+q)(x) \in \mathbb{W}$.

V6 - For any $t \in \mathbb{R}$ we have $(tp)(1) = tp(1) = t(0) = 0$, so $(tp)(x) \in \mathbb{W}$.

Consequently, $\mathbb{W}$ is a subspace of $P_3(\mathbb{R})$ by the Subspace Test.

Subspaces

Example 3

Let $\mathbb{S} = \{ax^2 + bx + c \in P_2(\mathbb{R}) \mid a^2 - b^2 = 0\}$. Is $\mathbb{S}$ a subspace of $P_2(\mathbb{R})$?

Solution

Observe that $x^2 + x$ and $2x^2 - 2x$ are in $\mathbb{S}$ since $1^2 - 1^2 = 0$ and $2^2 - (-2)^2 = 0$, but their sum
$(x^2 + x) + (2x^2 - 2x) = 3x^2 - x$ is not in $\mathbb{S}$ since $3^2 - (-1)^2 \neq 0$.
So, $\mathbb{S}$ is not closed under addition and hence is not a subspace.

Subspaces

Example 4

Let $\mathbb{S} = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \in M_{2\times2}(\mathbb{R}) \mid x_1 + x_2 = 2x_4 \right\}$. Is $\mathbb{S}$ a subspace of $M_{2\times2}(\mathbb{R})$?

Solution

By definition, $\mathbb{S}$ is a subset of $M_{2\times2}(\mathbb{R})$.
Also, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ satisfies $0 + 0 = 2(0)$, so $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{S}$.
Hence, $\mathbb{S}$ is non-empty.

Let $A, B \in \mathbb{S}$.
Then $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ where $a_1 + a_2 = 2a_4$, and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ where $b_1 + b_2 = 2b_4$.

V1 - We get $A + B = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix}$ where

$$(a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2) = 2a_4 + 2b_4 = 2(a_4 + b_4)$$

Thus, $A + B \in \mathbb{S}$.

V6 - We have $tA = \begin{bmatrix} ta_1 & ta_2 \\ ta_3 & ta_4 \end{bmatrix}$ where

$$ta_1 + ta_2 = t(a_1 + a_2) = t(2a_4) = 2(ta_4)$$

So, $tA \in \mathbb{S}$. Thus, $\mathbb{S}$ is a subspace of $M_{2\times2}(\mathbb{R})$ by the Subspace Test.