The Matrix Representation of a System of Linear Equations

When solving linear systems with Gaussian elimination and back-substitution, we can view the variables like placeholders.

So, if we keep our equations aligned properly, we can focus on the coefficients and forget about the variables when solving the system.

For example, the calculation

$$\begin{array}{rrcrcrcr} & -2x_1 & + & 3x_2 & - & 5x_3 & = & 7 \\ + & 2x_1 & + & x_2 & + & 3x_3 & = & -3 \\ \hline & & & 4x_2 & - & 2x_3 & = & 4 \end{array}$$

could be abbreviated as

$$\begin{array}{rrrrcr} & x_1 & x_2 & x_3 & & \\ \hline & -2 & 3 & -5 & = & 7 \\ + & 2 & 1 & 3 & = & -3 \\ \hline & 0 & 4 & -2 & = & 4 \end{array}$$

If it is assumed that the first column will always correspond to x_1, the second cloumn to x_2, etc., then the top row of variables is not necessary.

From now on, we will solve a system of linear equations by first writing all the coefficients in a rectangular array called a matrix.

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A general linear system of m equations in n unknowns will be represented by the matrix

$$\left[\begin{array}{cccccc|c} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} & b_i \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} & b_m \end{array}\right]$$

We have further abbreviated by replacing all of the equal signs with a vertical line.

The above matrix is known as the augmented matrix of the system.

If we remove the vertical line and the final column, we end up with the following coefficient matrix of the system:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

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Example

For the system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ -x_1 & + & 3x_2 & + & 4x_3 & = & 11 \\ 2x_1 & + & 5x_2 & + & x_3 & = & 3 \end{array}$$

The coefficient matrix is

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 4 \\ 2 & 5 & 1 \end{bmatrix}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ -1 & 3 & 4 & 11 \\ 2 & 5 & 1 & 3 \end{array}\right]$$

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Now, as each row in the matrix corresponds to an equation in our system, the Gaussian elimination steps we just learned become the following:

Definition: There are three types of elementary row operations (EROs) which correspond to the three steps of Gaussian elimination:

1. Multiply one row by a non-zero constant.
2. Interchange two rows.
3. Add a scalar multiple of one row to another row.

Definition: The process of performing elementary row operations on a matrix is called row reduction.

Definition: If a matrix M is row reduced into a matrix N by a sequence of elementary row operations, then we say that M is row equivalent to N, and we write $M \sim N$. Note that this is the same as saying that the corresponding systems of equations are equivalent.

Note:
Matrices and row reduction are *really* important concepts in this course.
It is imperative that you take your time working through this section of the course and learn to do related questions quickly and confidently.

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Notation

We notate the EROs as follows:

1. Multiplying row i by a non-zero constant c is notated as cR_i.
2. Interchanging rows i and j is notated as $R_i \updownarrow R_j$.
3. Adding c times row i to row j is notated as $R_j + cR_i$.

Additionally, we will use the symbol $\sim$ (read "equivalent") to indicate that two matrices are row equivalent.

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Example

Last lecture, we went through the Gaussian elimination steps to find the general solution to the system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ -x_1 & + & 3x_2 & + & 4x_3 & = & 11 \\ 2x_1 & + & 5x_2 & + & x_3 & = & 3 \end{array}$$

Now, we will do the exact same steps, but using matrix notation.

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ -1 & 3 & 4 & 11 \\ 2 & 5 & 1 & 3 \end{array}\right] R_2 + R_1 \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 5 & 3 & 7 \\ 2 & 5 & 1 & 3 \end{array}\right] R_3 - 2R_1 \sim$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 1 & 3 & 11 \end{array}\right] R_2 - 5R_3 \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 0 & -12 & -48 \\ 0 & 1 & 3 & 11 \end{array}\right]$$